Theories of Computation: Summative Assignment 3

This exercise is about context free grammars on the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. You may use the following facts that were mentioned in Week 3.

- Whether a given CFG accepts a given word is decidable.
- Whether the language of a given CFG is empty is decidable.
- Whether the language of a given CFG is Σ^* is undecidable.

Say that a CFG is *red* when it accepts every word of length 3 that begins with a, and *extremely red* when it accepts every word that begins with a.

1. Is redness decidable? Is it semidecidable? Explain your answer. **[3 marks]**
2. Is extreme redness decidable? Is it semidecidable? Explain your answer. (Hint: you may find it helpful to think about the negation of this property.) **[3 marks]**

Solution

1. Redness is decidable as we can simply check whether it accepts each of the words aaa, aab, aba, abb. If it accepts all of them, then it's red, and if not, then it isn't. Any decidable property is semidecidable, so redness is semidecidable.
2. Extreme redness is undecidable, for the following reason. Given a grammar G with start symbol S, we can form the grammar G' by adding a new start symbol S_0 with rule $S_0 ::= \mathtt{a}S$. The language of G is Σ^* iff G' is extremely red, so if extreme redness were decidable, we would have a decision procedure for whether the language of G is Σ^*, contradicting the fact stated above.

   The negation of extreme redness is semidecidable, for here is a semidecision procedure: given a grammar G, test if it accepts all words beginning with a of length 1, 2 etc. until we hit one that G doesn't accept, then we know that G is not extremely red, otherwise we will continue forever. If extreme redness were semidecidable, it would be decidable (since any semidecidable property with semidecidable negation is decidable), so extreme redness is not semidecidable.