Theories of Computation: Summative Assignment 2

To be handed in on Canvas before **Thursday 31st March, 5pm GMT**

Exercise 1 *Let $\Sigma = \{a, b\}$ and $u, v \in \Sigma^+$, where Σ^+ is the set of nonempty words over Σ. We say that u is* present *in v if u can be obtained by deleting letters from v. For example, abbba is present in aabbababa. We write $|u|$ to denote the* length *of word u, i.e., the number of characters. For example, $|abbba| = 5$.*

The goal of this exercise is to show that the following decision problem is in NP.

Input: *$w_0\#w_1\#...\#w_k$ such that $w_i \in \Sigma^+$ for $0 \leq i \leq k$.*

Problem: *is there a word $x \in \Sigma^+$ such that $|x| = |w_0|$ and x is present in each w_i for $0 < i \leq k$?*

We will decompose the task into several steps.

1. *Let us consider a two-tape deterministic Turing machine $\mathcal{M}_1$ on the input alphabet $\{a, b, \#\}$ with initial state 0, tape alphabet $T = \{a, b, \#, \llcorner\!\lrcorner\}$, return values $V = \{\texttt{True}, \texttt{False}\}$, and whose transition function is represented as the diagram in Figure 1 below.*

   Initially,

   - *the* Main *tape contains a non-empty block of as and bs (representing a word $w \in \Sigma^+$) in between a $\#$ on the left, on which the head is positioned, and a $\#$ or a $\llcorner\!\lrcorner$ on the right. (Outside of these $\#$s and/or $\llcorner\!\lrcorner$s there could be any symbol in T.)*
   - *the* Aux *tape contains a non-empty block of as and bs (representing a word $x \in \Sigma^+$) and is otherwise blank and the head is located on the $\llcorner\!\lrcorner$ immediately to the left.*

   For example,

Main	$\overset{\bullet}{\#}$	a	a	b	b	a	b	a	b	a	$\#$
Aux	$\overset{\bullet}{\llcorner\!\lrcorner}$	a	b	b	b	a	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$

   In the case where the input block on the Main *tape forms the word $w = a^n$ and the input block on the* Aux *tape forms the word $x = a^m$ for $m, n > 0$, how many steps does the machine $\mathcal{M}_1$ goes through until it returns a value in V? (Returning counts as a step.)* **[3 marks]**

This is in fact the worst case complexity as a function of $n = |w|$ and $m = |x|$ and you can use this fact in the remainder of the exercise.

2. *Design a two-tape nondeterministic Turing machine $\mathcal{M}_2$ that takes as an input a word $w \in \Sigma^+$ and can generate any word $x \in \Sigma^+$ that has the **same length** as w.*

   Formally, the start configuration is:

   - *the* Main *tape contains a nonempty block of as and bs (representing a word $w \in \Sigma^+$), with a $\llcorner\!\lrcorner$ to the left on which the head is placed and a $\#$ to the right, the rest of the tape is blank to the left and can contain any symbol in T beyond $\#$ on the right;*
   - *the* Aux *tape is blank.*

   For example,

Main	$\overset{\bullet}{\llcorner\!\lrcorner}$	a	a	b	b	a	b	a	b	a	$\#$
Aux	$\overset{\bullet}{\llcorner\!\lrcorner}$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$	$\llcorner\!\lrcorner$

   The machine $\mathcal{M}_2$ should stop when reaching a configuration where:

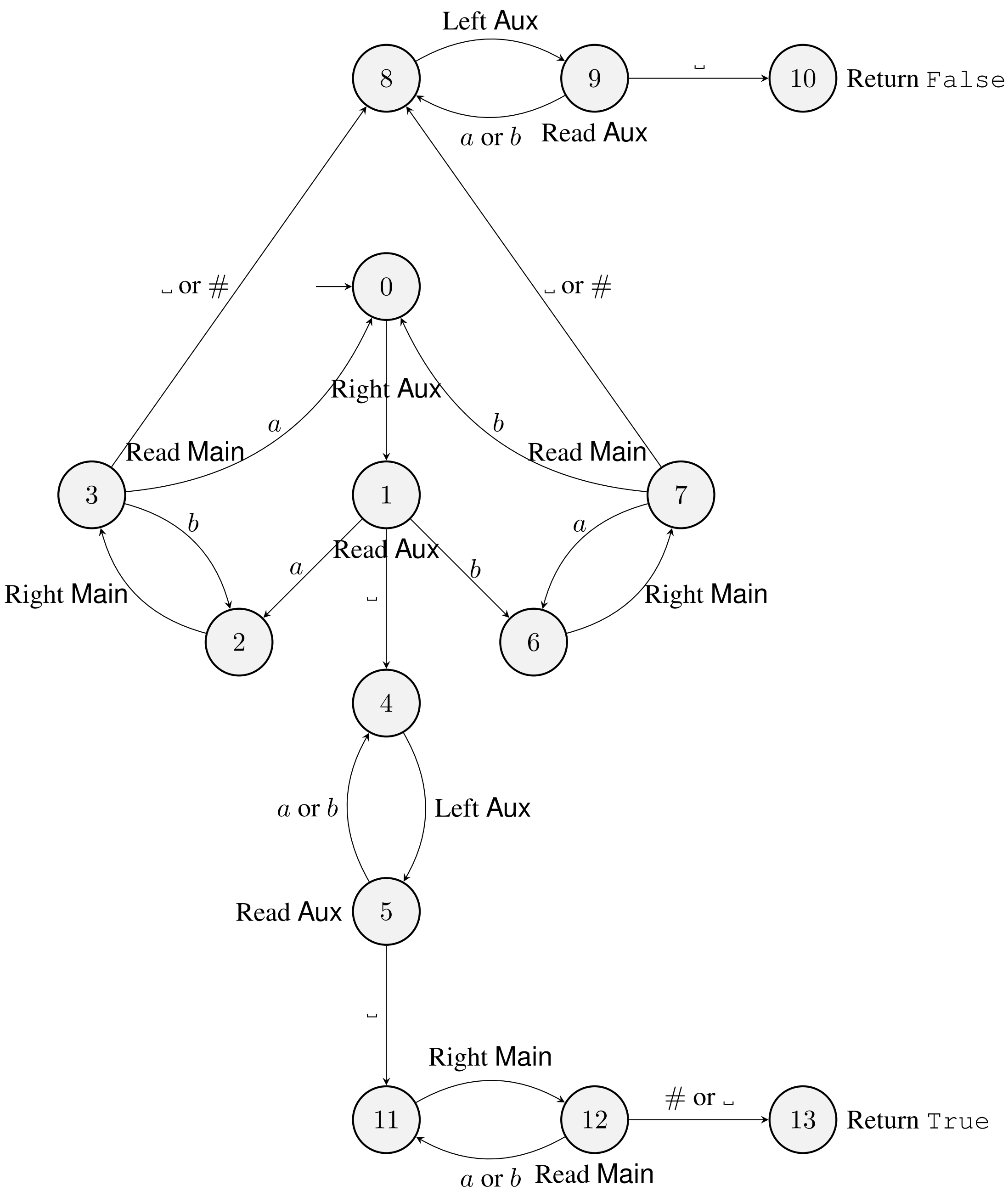

Figure 1: Transition diagram for machine $\mathcal{M}_1$

- *the Main tape is unchanged except for the head which should be placed on the $\#$ on the right*
- *the Aux tape contains an arbitrary block of as and bs (representing a word $x \in \Sigma^+$) of the same length as the input block on the Main tape and the head is on the first $\sqcup$ to the left.*

For example,

Main	$\sqcup$	a	a	b	b	a	b	a	b	a	$\overset{\bullet}{\#}$
Aux	$\overset{\bullet}{\sqcup}$	b	b	a	b	a	a	a	a	b	$\sqcup$

Give the machine $\mathcal{M}_2$ and briefly explain your solution. (Do not use more than 10 states.) **[3 marks]**

3. *Using machines $\mathcal{M}_1$ and $\mathcal{M}_2$ as macros, design a two-tape nondeterministic Turing machine $\mathcal{M}_3$ for the above decision problem.*

   This means that the machine should start with

   - *a block of as, bs and $\#$s representing the input $w_0\#w_1\#...\#w_k$ on an otherwise blank Main tape with the head on the first $\sqcup$ to the left of w_0*

- *and a blank* Aux *tape.*

The tape contents and head positions at the end do not matter.

Give the machine $\mathcal{M}_3$ *and briefly explain your solution. (Do not use more than 5 states.)* **[3 marks]**

4. *Explain briefly why the problem is in* NP. **[3 marks]**

   *(**Note:** You may assume that any polytime two-tape nondeterministic Turing machine can be converted into a polytime one-tape nondeterministic machine with the same language. Just as we learnt in lectures for deterministic machines.)*