Theories of Computation: Formative Assignment 2

To be handed in on Canvas before **Thursday 17th March, 5pm GMT**

Exercise 1 (Time Complexity in Big-O) *Let us consider two algorithms.*

Algorithm A_1 *has running time* $T_1(n) = \begin{cases} 5n^3 + 2 & for \quad 0 \leq n \leq 3 \\ 7n + 9 & for \quad n \geq 4 \end{cases}$

Algorithm A_2 *has running time* $T_2(n) = \begin{cases} 3n^4 + 3 & for \quad 0 \leq n \leq 2 \\ 2n^2 & for \quad n \geq 3 \end{cases}$

1. *Show that* $T_1(n)$ *is* $O(n)$ *and* $T_2(n)$ *is* $O(n^2)$. **[2 marks]**

   Remember: *To justify your claim that* $f(n)$ *is* $O(g(n))$ *provide constants* M *and* C *that satisfy the property that* $\forall n \geq M. f(n) \leq Cg(n)$.

2. *For each* $n \geq 0$ *which algorithm is more efficient? Justify your answer.* **[2 marks]**

Solution 1 *1.*

$T_1(n) = O(n)$*: For* $n \geq 4$, $T_1(n) = 7n + 9 \leq 10n$*. So we can take* $M_1 = 4$ *and* $C_1 = 10$.

$T_2(n) = O(n^2)$*: For* $n \geq 3$, $T_2(n) = 2n^2 \leq 2n^2$*. So we can take* $M_2 = 3$ *and* $C_2 = 2$.

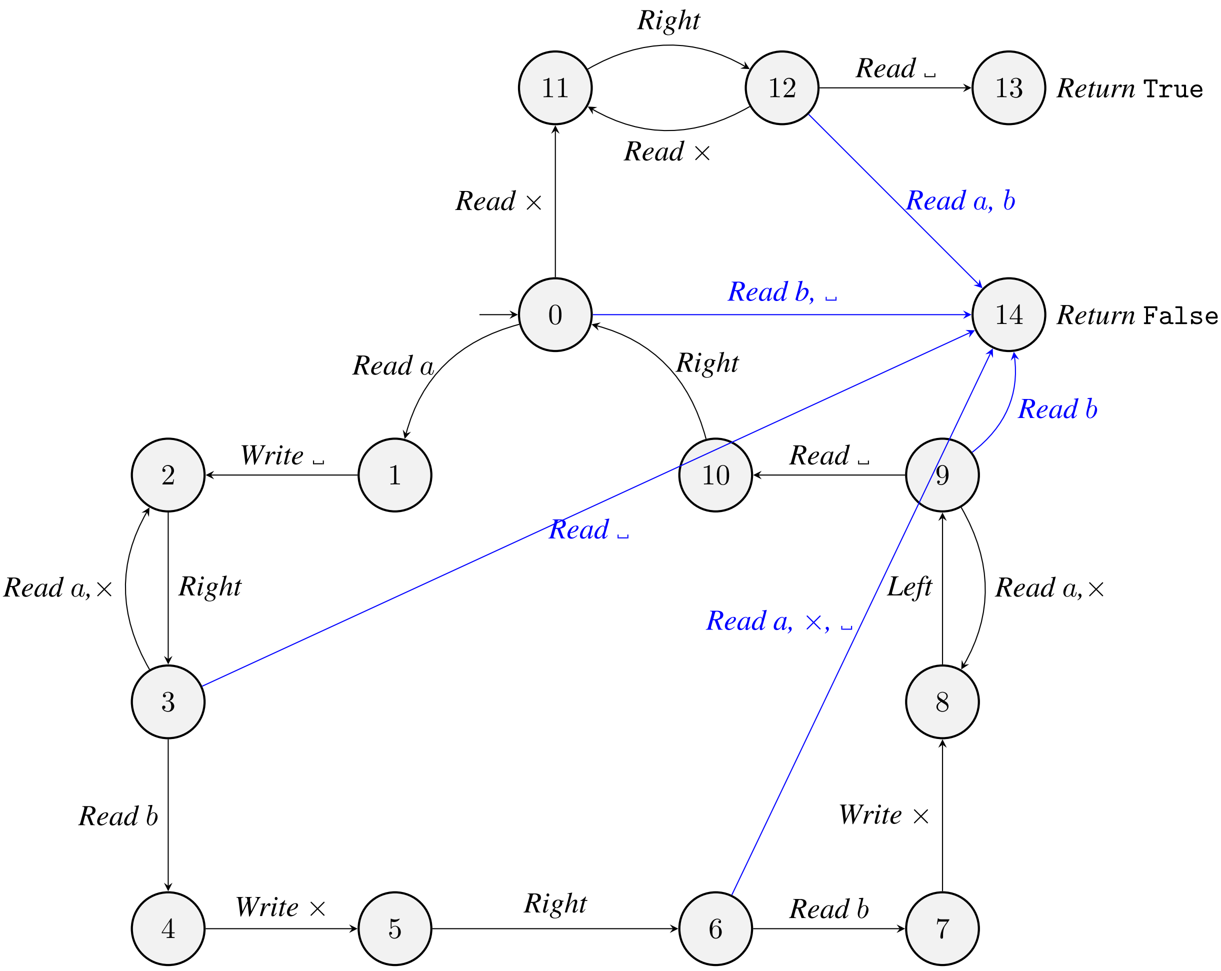
2.

n	0	1	2	3	4	5	6	7
$\mathbf{T_1(n)}$	2	7	42	137	37	44	51	58
$\mathbf{T_2(n)}$	3	6	51	18	32	50	72	98

For $n = 1$, $n = 3$ *and* $n = 4$, A_2 *is more efficient. For* $n = 0$, $n = 2$ *and* $n \geq 5$, A_1 *is more efficient.*

Indeed, when $n \geq 4$, $T_1(n) = T_2(n) \Leftrightarrow 2n^2 - 7n - 9 = 0 \Leftrightarrow n = 4.5$.

Exercise 2 (Turing Machines) *Let us consider a Turing machine M on the input alphabet $\Sigma = \{a, b\}$ with initial state 0, tape alphabet $T = \{a, b, \textvisiblespace, \times\}$, return values $V = \{\texttt{True}, \texttt{False}\}$, and whose transition function is represented as the following diagram.*



Initially the tape contains a non-empty block of as and bs and is otherwise blank. The head is positioned on the first non blank character.

1. *Give the complete run of the machine M above on the word ab. At each step, indicate the tape contents, the position of the head, the current state and the instruction (including the result if it is a Read).* **[2 marks]**

   Hint: *No more than 10 steps are needed.*

2. *Without justification, does M accept words abb and $abbb$ (i.e. return* `True` *if given them as input)?* **[2 marks]**

3. *What is the language $\mathcal{L}(M)$ recognised by M?* **[2 marks]**

4. *Use machine M as a macro and design a Turing machine with five states that recognises the language $\mathcal{L} = \{a^{n+1}b^{2n} \mid n \geq 1\}$?* **[2 marks]**

Solution 2 *1. M does not accept word ab.*

$\sqcup$	$\overset{\bullet}{a}$	b	$\sqcup$	*0*	*Read a*
$\sqcup$	$\overset{\bullet}{a}$	b	$\sqcup$	*1*	*Write* $\sqcup$
$\sqcup$	$\overset{\bullet}{\sqcup}$	b	$\sqcup$	*2*	*Right*
$\sqcup$	$\sqcup$	$\overset{\bullet}{b}$	$\sqcup$	*3*	*Read b*
$\sqcup$	$\sqcup$	$\overset{\bullet}{b}$	$\sqcup$	*4*	*Write* $\times$
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	$\sqcup$	*5*	*Right*
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{\sqcup}$	*6*	*Read* $\sqcup$
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{\sqcup}$	*14*	*Return* `False`

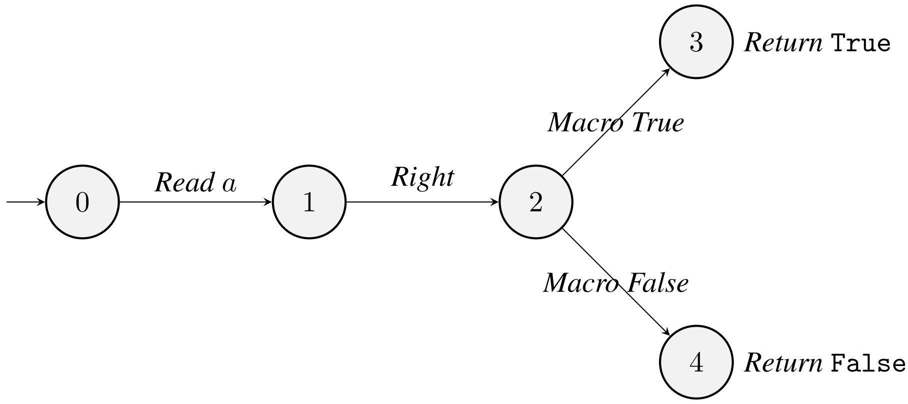
2. M accepts word abb but does not accept word $abbb$. (Full runs below were not expected as answers.)

$\sqcup$	$\overset{\bullet}{a}$	b	b	$\sqcup$	*0*	*Read a*
$\sqcup$	$\overset{\bullet}{a}$	b	b	$\sqcup$	*1*	*Write* $\sqcup$
$\sqcup$	$\overset{\bullet}{\sqcup}$	b	b	$\sqcup$	*2*	*Right*
$\sqcup$	$\sqcup$	$\overset{\bullet}{b}$	b	$\sqcup$	*3*	*Read b*
$\sqcup$	$\sqcup$	$\overset{\bullet}{b}$	b	$\sqcup$	*4*	*Write* $\times$
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	b	$\sqcup$	*5*	*Right*
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{b}$	$\sqcup$	*6*	*Read b*
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{b}$	$\sqcup$	*7*	*Write* $\times$
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{\times}$	$\sqcup$	*8*	*Left*
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	$\times$	$\sqcup$	*9*	*Read* $\times$
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	$\times$	$\sqcup$	*8*	*Left*
$\sqcup$	$\overset{\bullet}{\sqcup}$	$\times$	$\times$	$\sqcup$	*9*	*Read* $\sqcup$
$\sqcup$	$\overset{\bullet}{\sqcup}$	$\times$	$\times$	$\sqcup$	*10*	*Right*
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	$\times$	$\sqcup$	*0*	*Read* $\times$
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	$\times$	$\sqcup$	*11*	*Right*
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{\times}$	$\sqcup$	*12*	*Read* $\times$
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{\times}$	$\sqcup$	*11*	*Right*
$\sqcup$	$\sqcup$	$\times$	$\times$	$\overset{\bullet}{\sqcup}$	*12*	*Read* $\sqcup$
$\sqcup$	$\sqcup$	$\times$	$\times$	$\overset{\bullet}{\sqcup}$	*13*	*Return* `True`

$\sqcup$	$\overset{\bullet}{a}$	b	b	b	$\sqcup$	*0*	*Read a*
$\sqcup$	$\overset{\bullet}{a}$	b	b	b	$\sqcup$	*1*	*Write* $\sqcup$
$\sqcup$	$\overset{\bullet}{\sqcup}$	b	b	b	$\sqcup$	*2*	*Right*
$\sqcup$	$\sqcup$	$\overset{\bullet}{b}$	b	b	$\sqcup$	*3*	*Read b*
$\sqcup$	$\sqcup$	$\overset{\bullet}{b}$	b	b	$\sqcup$	*4*	*Write* $\times$
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	b	b	$\sqcup$	*5*	*Right*
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{b}$	b	$\sqcup$	*6*	*Read b*
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{b}$	b	$\sqcup$	*7*	*Write* $\times$
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{\times}$	b	$\sqcup$	*8*	*Left*
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	$\times$	b	$\sqcup$	*9*	*Read* $\times$
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	$\times$	b	$\sqcup$	*8*	*Left*
$\sqcup$	$\overset{\bullet}{\sqcup}$	$\times$	$\times$	b	$\sqcup$	*9*	*Read* $\sqcup$
$\sqcup$	$\overset{\bullet}{\sqcup}$	$\times$	$\times$	b	$\sqcup$	*10*	*Right*
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	$\times$	b	$\sqcup$	*0*	*Read* $\times$
$\sqcup$	$\sqcup$	$\overset{\bullet}{\times}$	$\times$	b	$\sqcup$	*11*	*Right*
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{\times}$	b	$\sqcup$	*12*	*Read* $\times$
$\sqcup$	$\sqcup$	$\times$	$\overset{\bullet}{\times}$	b	$\sqcup$	*11*	*Right*
$\sqcup$	$\sqcup$	$\times$	$\times$	$\overset{\bullet}{b}$	$\sqcup$	*12*	*Read b*
$\sqcup$	$\sqcup$	$\times$	$\times$	$\overset{\bullet}{b}$	$\sqcup$	*14*	*Return* `False`

3. $\mathcal{L}(M) = \{a^n b^{2n} \mid n \geq 1\}$

4. Using machine M as a macro instruction at state 2, we can design the following machine to recognise $\mathcal{L}$:


0
Read a
1
Right
2
Macro True
Macro False
3
Return True
4
Return False