Theories of Computation

Solutions

May 2022

Theories of Computation

Answer ALL questions. The paper will be marked out of 60, which will be rescaled to a mark out of 100.

Exam paper

Question 1 : Regular Languages and Automata

Consider the regular expression $E = (b \mid ab)^*(a \mid \varepsilon)$ on alphabet $\Sigma = \{a, b\}$.

(a) Do the following words match E? Explain your answer.

- (i) ε
- (ii) $abba$
- (iii) aaa

[6 marks]

(b) Give a minimal total DFA that recognizes the language described by E and prove that it is minimal.

[9 marks]

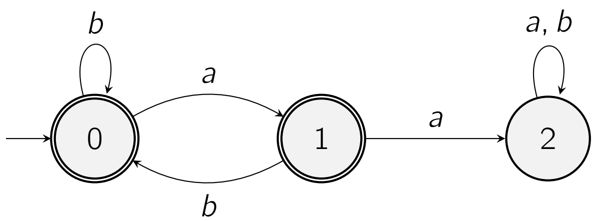
Solutions

(a) (i) Yes, because we can have zero instances of the first part i.e. $(b \mid ab)^*$ and choose ε from the second part i.e. $(a \mid \varepsilon)$ to match with ε

(ii) Yes, because we can obtain ab, followed by b from the first part i.e. $(b \mid ab)^*$ and choose a from the second part i.e. $(a \mid \varepsilon)$ to match with $abba$

(iii) No, because we cannot obtain any instances of a alone from the first part i.e. $(b \mid ab)^*$ and the second part i.e. $(a \mid \varepsilon)$ only allows to choose a single instance of a or ε. Therefore, we cannot match aaa using this regular expression.

(b) The minimal total DFA is shown below:



The DFA is minimal. Proof is given below:
We can see that each state is reachable using the following inputs:

- state 0 is reachable on ε
- state 1 is reachable on a
- state 2 is reachable on aa

and all the states are pairwise inequivalent:

- 0 and 1 are not equivalent because 0 accepts a and 1 does not;
- 0 and 2 are not equivalent because 0 accepts ε and 2 does not;
- 1 and 2 are not equivalent because 1 accepts ε and 2 does not.

Question 2 : Context-free Languages

Consider the following context-free grammar $\mathcal{G}$ on the alphabet $\Sigma = \{a, b\}$

$$\begin{array}{rrcl} \Rightarrow & S & ::= & XX \\ & X & ::= & aXa \mid bXb \mid a \mid b \mid \varepsilon \end{array}$$

(a) Show that the grammar $\mathcal{G}$ is ambiguous. **[7 marks]**

(b) A student is in the process of transforming $\mathcal{G}$ into Chomsky Normal Form and has reached the following:

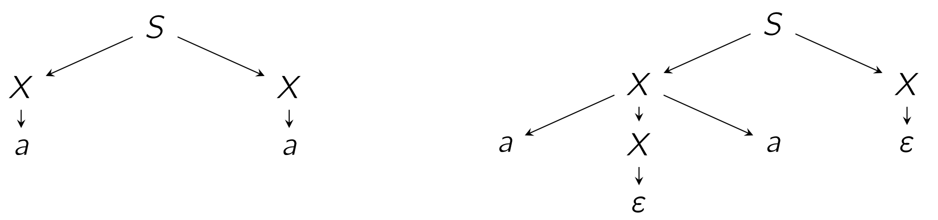
$$\begin{array}{rrcl} \Rightarrow & S_0 & ::= & S \\ & S & ::= & XX \\ & X & ::= & AU \mid BV \mid a \mid b \mid \varepsilon \\ & U & ::= & XA \\ & A & ::= & a \\ & V & ::= & XB \\ & B & ::= & b \end{array}$$

The student's next step is to remove the rule $X ::= \varepsilon$. Give the grammar that results from this step.

[8 marks]

Solutions

(a) $\mathcal{G}$ is ambiguous since the word aa has (at least) two possible derivation trees in this grammar.

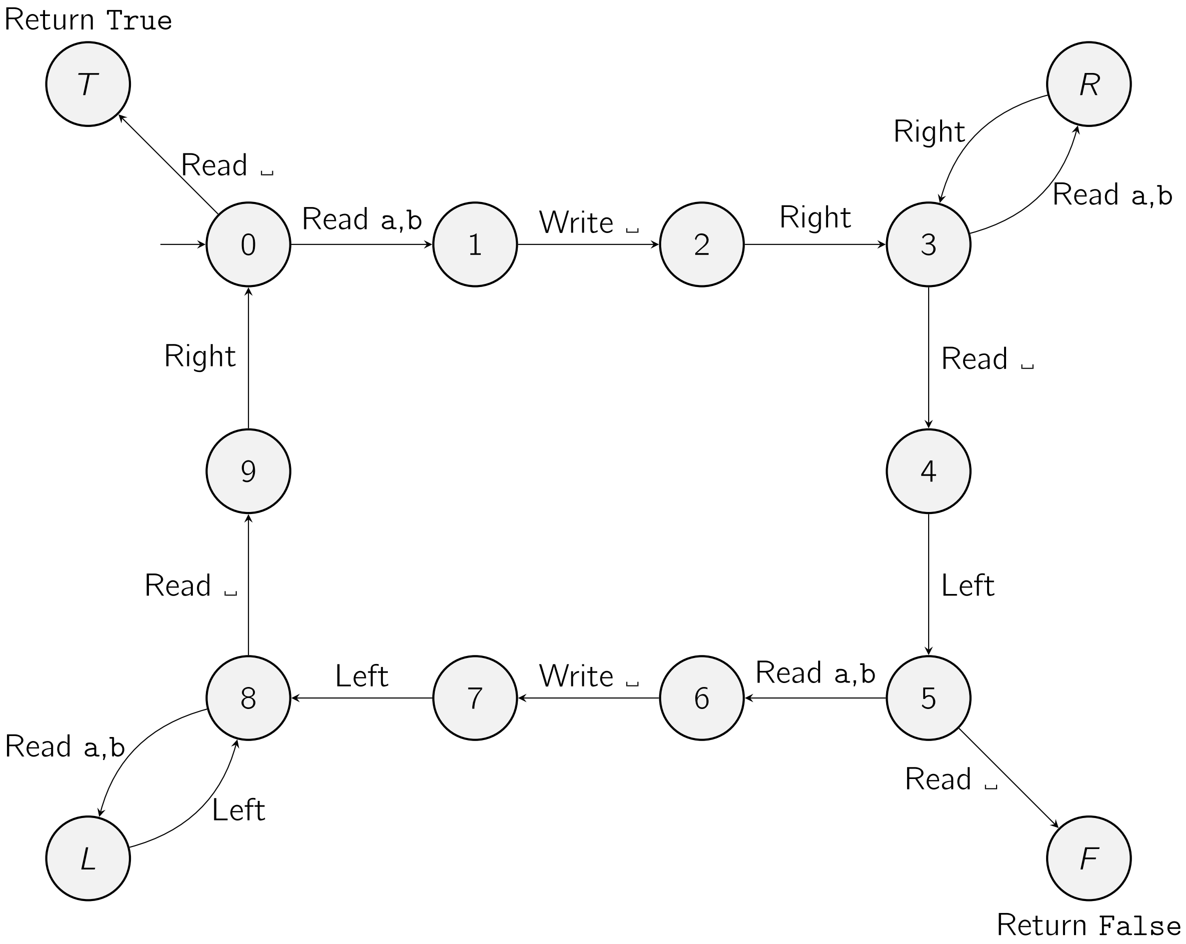


(b)

$$\begin{array}{rrcl} \Rightarrow & S_0 & ::= & S \\ & S & ::= & XX \mid X \mid \varepsilon \\ & X & ::= & AU \mid BV \mid a \mid b \\ & U & ::= & XA \mid A \\ & A & ::= & a \\ & V & ::= & XB \mid B \\ & B & ::= & b \end{array}$$

Question 3 : Turing Machines and Complexity

Consider the following deterministic Turing machine $\mathcal{M}$ on alphabet $\Omega = \{\mathtt{a}, \mathtt{b}, \llcorner\!\lrcorner\}$. The tape initially contains a nonempty block of a's and b's on an otherwise blank tape with the head on the leftmost character. The transition function is given by the following diagram:



(a) Trace the behaviour of the machine $\mathcal{M}$ on the word aa. **[7 marks]**

(b) Recall the notation $\sum_{k=0}^{p} x_k$ for $x_0 + x_1 + \cdots + x_p$.

The processing time for a block of length $n > 0$ is as follows.

- In the case where $n = 2p+2$ ($p \geq 0$) the number of steps is $(\sum_{k=0}^{p}(8k+12))+2$.
- In the case where $n = 2p+1$ ($p \geq 0$) the number of steps is $(\sum_{k=0}^{p}(8k+8))-1$.

Show that the complexity of $\mathcal{M}$ is in $O(n^2)$.

[8 marks]

Solutions

(a)

1	␣	• a	a	␣	␣	0	Read a
2	␣	• a	a	␣	␣	1	Write ␣
3	␣	• ␣	a	␣	␣	2	Right
4	␣	␣	• a	␣	␣	3	Read a
5	␣	␣	• a	␣	␣	R	Right
6	␣	␣	a	• ␣	␣	3	Read ␣
7	␣	␣	a	• ␣	␣	4	Left
8	␣	␣	• a	␣	␣	5	Read a
9	␣	␣	• a	␣	␣	6	Write ␣
10	␣	␣	• ␣	␣	␣	7	Left
11	␣	• ␣	␣	␣	␣	8	Read ␣
12	␣	• ␣	␣	␣	␣	9	Right
13	␣	␣	• ␣	␣	␣	0	Read ␣
14	␣	␣	• ␣	␣	␣	T	Return `True`

(b)

- in the case where $n = 2p+2$ ($p \geq 0$) the number of steps is $\sum_0^p (8k+12)+2$ $= 8\frac{p(p+1)}{2}+12(p+1)+2 = 4p^2+16p+14 = 4(\frac{n-2}{2})^2+16(\frac{n-2}{2})+14 = n^2+4n+2$
- in the case where $n = 2p+1$ ($p \geq 0$) the number of steps is $\sum_0^p (8k+8)-1$ $= 8\frac{p(p+1)}{2}+8(p+1)-1 = 4p^2+12p+7 = 4(\frac{n-1}{2})^2+12(\frac{n-1}{2})+7 = n^2+4n+2$

In conclusion the complexity of $\mathcal{M}$ as a function of the length n of the input is $T(n) = n^2+4n+2$.

For $n \geq 1$, $n^2+4n+2 \leq 7n^2$, so by taking $M = 1$ and $C = 7$ we have that $T(n) \in O(n^2)$.

Question 4 : Models of Computation and Decidability

(a) *Removed. Not on the syllabus this year.* **[7 marks]**

(b) A program in Java is said to be *purple* if it either halts or contains (in the body code) an even number of `a`'s. Show that purpleness is undecidable.

[8 marks]

Solutions

(a) *Removed. Not on the syllabus this year.*

(b) Suppose for a contradiction that the property $P(c)$: "program c halts or contains (in the body code) an even number of `a`'s" is decidable. This means that there exists a Turing machine such that given as input the encoding of the code of a program c answers `Yes` when $P(c)$ is true and `No` when $P(c)$ is false.

We will show that we can use this Tester to decide the Halting problem.

A program c can either

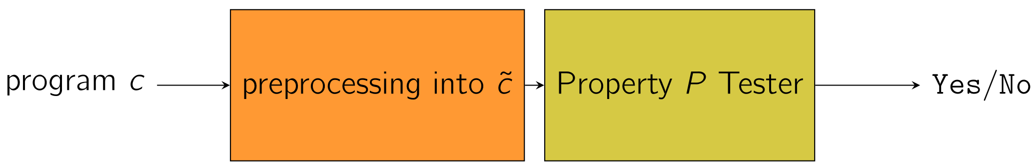
(i) halt

(ii) not halt and have an even number of `a`'s

(iii) not halt and have an odd number of `a`'s

In the first two cases $P(c)$ would be true and in the last one it would be false.

We add to the Tester a preprocessing step which counts the number of `a`'s in the code of c and transform c into $\tilde{c} = c$ with a comment `/* a */` at the end in the case where c has an even number of `a`'s or $\tilde{c} = c$ otherwise. Therefore $\tilde{c}$ halts if and only if c halts but always has an odd numbers of `a`'s, meaning that we can only be in case (i) or (iii).



The combination of the preprocessing step and the testing step will answer `Yes` when c halts and `No` when c does not halt. This contradicts the undecidability of the Halting problem. Hence, property P is undecidable.