Mathematical and Logical Foundations of Computer Science
Logic Cheat Sheet

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1 Propositional Logic

1.1 Syntax

The syntax of propositional logic formulas is defined by the following grammar, where a ranges over atomic propositions (atoms):

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

There are two special atoms: $\top$ which stands for True, and $\bot$ which stands for False.
We use the following precedence and associativity rules:

- Precedence: in decreasing order of precedence $\neg, \wedge, \vee, \rightarrow$.
- Associativity: all operators are right associative

1.2 Constructive Natural Deduction

$$\frac{\bot}{A}\ [\bot E] \qquad \frac{}{\top}\ [\top I] \qquad \frac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ B\end{array}}{A \rightarrow B}\ 1\ [\rightarrow I] \qquad \frac{A \rightarrow B \quad A}{B}\ [\rightarrow E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ \bot\end{array}}{\neg A}\ 1\ [\neg I] \qquad \frac{\neg A \quad A}{\bot}\ [\neg E]$$

$$\frac{A}{A \vee B}\ [\vee I_L] \qquad \frac{A}{B \vee A}\ [\vee I_R] \qquad \frac{A \vee B \quad A \rightarrow C \quad B \rightarrow C}{C}\ [\vee E]$$

$$\frac{A \quad B}{A \wedge B}\ [\wedge I] \qquad \frac{A \wedge B}{B}\ [\wedge E_R] \qquad \frac{A \wedge B}{A}\ [\wedge E_L]$$

1.3 Classical Natural Deduction

It includes all the Constructive Natural Deduction rules, plus:

$$\frac{}{A \vee \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

1.4 Truth Tables

A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

A	B	$A \wedge B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

A	B	$A \rightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

A	$\neg A$
T	**F**
F	**T**

$\top$
T

$\bot$
F

2 Predicate Logic

2.1 Syntax

The syntax of predicate logic is defined by the following grammar:

$$\begin{array}{lcl} t & ::= & x \mid f(t,\ldots,t) \\ P & ::= & p(t,\ldots,t) \mid \neg P \mid P \land P \mid P \lor P \mid P \to P \mid \forall x.P \mid \exists x.P \end{array}$$

There are two special predicate symbols of arity 0: $\top$ which stands for True, and $\bot$ which stands for False.

2.2 Natural Deduction

The Natural Deduction rules for Predicate Logic include all Proposition Logic rules plus the following rules:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \qquad \begin{array}{c}\overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q\end{array}}{Q}\ 1\ [\exists E]$$

Side conditions:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists I]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

2.3 Semantics

Given a signature: $\langle\langle f_1^{k_1},\ldots,f_n^{k_n}\rangle,\langle p_1^{j_1},\ldots,p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \le i \le n$
- of predicate symbols p_i of arity j_i, for $1 \le i \le m$

a model M is a structure $\langle D,\langle \mathcal{F}_{f_1},\ldots,\mathcal{F}_{f_n}\rangle,\langle \mathcal{R}_{p_1},\ldots,\mathcal{R}_{p_m}\rangle\rangle$

1. of a non-empty domain D
2. interpretations $\mathcal{F}_{f_i} \in D^{k_i} \to D$ for function symbols f_i
3. interpretations $\mathcal{R}_{p_i} \subseteq D^{j_i}$ for predicate symbols p_i

A variable valuation v is a partial function from variables to D

Given a model M and a variable valuation v, we assign meaning to terms and formulas as follows:

- Meaning of terms:
  - $[\![x]\!]_v^M = v(x)$
  - $[\![f(t_1,\ldots,t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M,\ldots,[\![t_n]\!]_v^M\rangle)$
- Meaning of formulas:
  - $\models_{M,v} \top$ is true
  - $\models_{M,v} \bot$ is false
  - $\models_{M,v} p(t_1,\ldots,t_n)$ iff $\langle [\![t_1]\!]_v^M,\ldots,[\![t_n]\!]_v^M\rangle \in \mathcal{R}_p$
  - $\models_{M,v} \neg P$ iff $\neg\models_{M,v} P$
  - $\models_{M,v} P \land Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
  - $\models_{M,v} P \lor Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
  - $\models_{M,v} P \to Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
  - $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x\mapsto d)} P$
  - $\models_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\models_{M,(v,x\mapsto d)} P$