Maths Exercise Revisions

① **Base case**: $n=0 \longrightarrow 5n^3+n=0 \equiv 0[6]$ ✓

Inductive step: Suppose for a given $n \in \mathbb{N}$, $5n^3+n \equiv 0[6]$

$$\underbrace{5(n+1)^3}+(n+1) = 5n^3+15n^2+15n+5+n+1 = \underbrace{5n^3+n}_{\equiv 0[6]}+15n^2+15n+6$$

$$(n+1)(n+1)^2 = (n+1)(n^2+2n+1)$$
$$= n^3+2n^2+n+n^2+2n+1 = n^3+3n^2+3n+1$$

$$\equiv 15\underbrace{n(n+1)}\,[6] \equiv 0[6] \checkmark$$

one of n and n+1 is even

suppose wlog $n=2k$

Conclusion: We have proved by induction that for all $n \in \mathbb{N}$: $5n^3+n \equiv 0[6]$.

② (a) $x = 12 \qquad y = -112 \qquad z = 119$

(b)
$$\begin{array}{rl} & \overset{1\,1}{0000} \quad \overset{1}{1}100 \\ + & 0111 \quad 0111 \\ \hline & 1000 \quad 0011 \end{array}$$

encodes -125 as opposed to $x+y=131$

encoding modulo $2^8 = 256$

$\longrightarrow$ overflow!

③ (a) $x = \boxed{0}\boxed{111}\boxed{0000}$ $\qquad x = 2^4$

sign +

mantissa $m = 1$

exponent $e = 7-(2^{3-1}-1) = 4$

$y = \boxed{1}\boxed{001}\boxed{0001}$ $\qquad y = 1.0625 \times 2^{-2}$

sign −

mantissa $m = 1 + \frac{1}{2^4} = 1.0625$

exponent $e = 1-(2^{3-1}-1) = -2$

(b) $a = 2.5 = 1.25 \times 2^1 \qquad a = 0100\ 0100$

$b = -1.125 = -1.125 \times 2^0 \qquad b = 1011\ 0010$

(c) $g = 1.03125 = \left(1+\frac{1}{2^5}\right)\times 2^0$

not possible to represent on 4bit-mantissa
will be approximated as

$$g = \underline{0011\ 0011} \rightarrow \left(1+\frac{1}{2^4}\right)\times 2^0$$

$h = 48 = 1.5 \times 2^5$

not possible to represent with 3bit-exponent
will cause overflow on exponent

$$h = \underline{0100\ 1000} \rightarrow 1.5 \times 2^{-3}$$

④ $A = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 100\} \subseteq \mathbb{R}$

$B = \{n \in \mathbb{N} \mid n = p^3 \text{ for } p \in \mathbb{N}\} \subseteq \mathbb{N}$

$C = \{a, b\}$

$\subseteq \mathbb{N}$

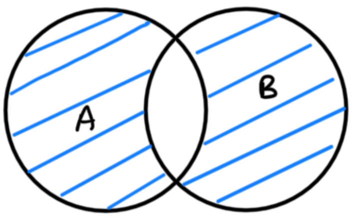
$A \cap B = \{0, 1, 8, 27, 64\}$

$(A \cap B) \times C$

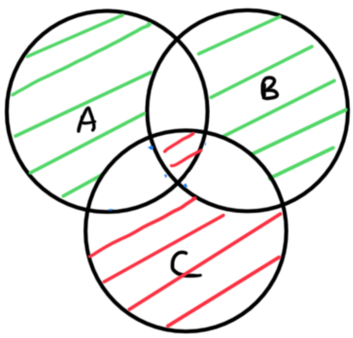
intersection product

$= \{(0,a), (0,b), (1,a), (1,b), (8,a), (8,b), (27,a), (27,b), (64,a), (64,b)\}$

⑤



for $A \blacktriangledown B$



for $\left(\underline{(A \blacktriangledown B) \setminus C}\right) \cup \left(\underline{C \setminus (A \blacktriangledown B)}\right)$

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- <u>reflexive:</u> $x = 1.x$ and $y = 1.y \implies ((x,y),(x,y)) \in R$
- <u>symmetric:</u> suppose $((x,y),(x',y')) \in R$

  hence $x' = ax$ and $y' = by$ for some $a,b > 0$

  so $x = \frac{1}{a}x'$ and $y = \frac{1}{b}y' \implies ((x',y'),(x,y)) \in R$
- <u>transitive:</u> suppose $((x,y),(x',y')) \in R$ and $((x',y'),(x'',y'')) \in R$

  hence $x' = ax$, $y' = by$, $x'' = cx'$, and $y'' = dy'$ for some $a,b,c,d > 0$

  so $x'' = acx$ and $y'' = bdy \implies ((x,y),(x'',y'')) \in R$

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- <u>reflexive:</u> $x \leqslant x$ and $y \leqslant y \implies ((x,y),(x,y)) \in R$
- <u>transitive:</u> suppose $((x,y),(x',y')) \in R$ and $((x',y'),(x'',y'')) \in R$

  hence $x \leqslant x'$, $y \leqslant y'$, $x' \leqslant x''$ and $y' \leqslant y''$

  so $x \leqslant x''$ and $y \leqslant y'' \implies ((x,y),(x'',y'')) \in R$
- <u>antisymmetric:</u> suppose $((x,y),(x',y')) \in R$ and $((x',y'),(x,y)) \in R$

  hence $x \leqslant x'$, $y \leqslant y'$, $x' \leqslant x$ and $y' \leqslant y$

  so $x = x'$ and $y = y' \implies (x,y) = (x',y')$

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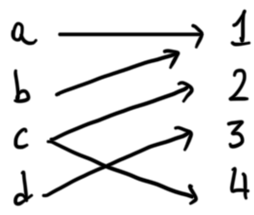
(a) Draw a diagram of the situation



(b) Is R a function from A to B? No because c has two images

(c) Take $B' = \{1,2,3\} \subseteq B$. Is $R \cap (A \times B')$ a function? Is it injective?

Yes but not injective because a and b have same image

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- <u>injective</u>: suppose $f(n,m) = f(n',m')$ i.e. $nm = n'm'$

  then (n,m) is not necessarily $= (n',m')$

  take for example $n = m' = 3$ and $n' = m = 2$. $\Longrightarrow$ No

- <u>surjective</u>: need to show that for any $p \in \mathbb{N}$

  there exists $(n,m) \in \mathbb{N}^2$ such that $nm = p$

  enough to take $n = 1$ and $m = p$ $\Longrightarrow$ YES

10 (a) $A_n = \{0, 1, 2 \dots n\}$ is finite for any $n \geqslant 1$.

(b) We have seen in class that $B_1 = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$ is uncountable
And for any $n \geqslant 1$ $B_1 \subseteq B_n$
So B_n is also uncountable.