School of Computer Science

First Year Undergraduate

06-35324
LC Mathematical and Logical Foundations of Computer Science

Main Summer Examinations 2025

Time allowed: 2 hours

[Answer all questions]

Note

Answer ALL questions. There are 4 questions. Questions 1 and 2 are worth 30 marks each, and questions 3 and 4 are worth 20 marks each.

Question 1 [Numbers & Sets – 30 marks]

(a) Consider the sequence $(a_n)_n$ of integers given by

- $a_0 = 52$,
- $a_{n+1} = 5a_n + 2$.

Prove by induction that, for all $n \in \mathbb{N}$,

$$a_n \equiv 3 \pmod 7.$$

Show your calculations and explain each step of your proof carefully. **[10 marks]**

(b) (i) Perform Euclid's algorithm on the numbers 7 and 12. Detail your process including giving the values of the sequence $r_0, r_1, \ldots$. **[4 marks]**

(ii) By finding Bézout coefficients, find all integers $x \in \mathbb{Z}$ satisfying both

$$x \equiv 1 \pmod 7 \quad \text{and} \quad x \equiv 2 \pmod{12}.$$

[6 marks]

(c) Consider the set A given by

$$A = \{x \in \mathbb{R} \mid (x-1)^2 \in \mathbb{Q}\},$$

i.e., A is the set of all real numbers x such that $(x-1)^2$ is a rational number. In the following, you may use standard facts about irrational numbers without proof provided they are stated clearly.

(i) Show that $\mathbb{Q} \subseteq A$. **[3 marks]**

(ii) Show that $\sqrt{2} + 1 \in A$. **[2 marks]**

(iii) Show that $\sqrt{2} \notin A$. **[2 marks]**

(iv) Show that $A \cap [1, \infty)$ is countable by defining an injection $A \cap [1, \infty) \to \mathbb{Q}$. **[3 marks]**

Question 2 [Linear Algebra – 30 marks]

Throughout this question, the field is complex numbers (with usual addition and multiplication).

(a) (i) State the definition of $\vec{x}$ and α being eigenvector and eigenvalue of a matrix A. **[3 marks]**

(ii) Determine the eigenvectors and eigenvalues of the following matrices (show all your working and justification)

$$A_1 = \begin{pmatrix} -1 & 0 \\ 4 & -2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

[10 marks]

(b) Find the determinant of the following matrix (show all your working and justification)

$$A_3 = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 2 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

[3 marks]

(c) Find the inverse of the following matrix, using Gaussian Elimination (show all your working)

$$A_4 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ -3 & -3 & 1 \end{pmatrix}$$

[8 marks]

(d) (i) The Invertible Matrix theorem contains several equivalent statements for a matrix to be invertible. List four of these statements. **[4 marks]**

(ii) Are the two following matrices invertible? Justify your answer.

$$A_5 = \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}, \qquad A_6 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

[2 marks]

Question 3 [Propositional Logic – 20 marks]

Let G be the formula $\neg(A \land B) \to B \to \neg A$, and
let H be the formula $(A \lor B) \to (B \to A) \to (A \to \neg B) \to \neg(A \to B)$.
(See the logic appendix below for the Natural Deduction rules for Propositional Logic.)

(a) Provide parse trees corresponding to G and H. **[4 marks]**

(b) Provide a constructive Natural Deduction proof of G. **[4 marks]**

(c) Provide a constructive Natural Deduction proof of H. **[8 marks]**

(d) Provide: (i) an example of a formula that is both satisfiable and falsifiable, and (ii) an example of a formula that is valid. Both formulas should use the variables A and B and no others. **[4 marks]**

Question 4 [Predicate Logic – 20 marks]

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: `even` (arity 1); `odd` (arity 1)

For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc. *(See the logic appendix below for the Natural Deduction rules for Predicate Logic.)*

(a) Provide a constructive Natural Deduction proof of:

$$\exists x.\mathtt{even}(x) \to (\exists x.\mathtt{even}(x) \lor \mathtt{odd}(x))$$

[6 marks]

(b) Provide a constructive Natural Deduction proof of:

$$\mathtt{even}(0) \to (\forall x.\mathtt{even}(x) \to \mathtt{even}(\mathtt{succ}(\mathtt{succ}(x)))) \to \mathtt{even}(2)$$

[6 marks]

(c) The model $\langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \emptyset, \emptyset \rangle \rangle$ does not satisfy the formula $\exists x.\mathtt{even}(x)$. Modify the above model so that it satisfies $\exists x.\mathtt{even}(x)$. **[4 marks]**

(d) Provide a model that satisfies $\forall x.\mathtt{even}(x)$. **[4 marks]**

End of Questions – See below for the Logic appendix

Logic appendix

1 Propositional Logic

1.1 Syntax

The syntax of propositional logic formulas is defined by the following grammar, where a ranges over atomic propositions (atoms):

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \to P \mid \neg P$$

There are two special atoms: $\top$ which stands for True, and $\bot$ which stands for False. We use the following precedence and associativity rules:

- Precedence: in decreasing order of precedence $\neg, \wedge, \vee, \to$.
- Associativity: all operators are right associative

1.2 Constructive Natural Deduction

$$\frac{\bot}{A}\ [\bot E] \qquad \frac{}{\top}\ [\top I] \qquad \frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots \\ B\end{array}}{A \to B}\ 1\ [\to I] \qquad \frac{A \to B \quad A}{B}\ [\to E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots \\ \bot\end{array}}{\neg A}\ 1\ [\neg I] \qquad \frac{\neg A \quad A}{\bot}\ [\neg E]$$

$$\frac{A}{A \vee B}\ [\vee I_L] \qquad \frac{A}{B \vee A}\ [\vee I_R] \qquad \frac{A \vee B \quad A \to C \quad B \to C}{C}\ [\vee E]$$

$$\frac{A \quad B}{A \wedge B}\ [\wedge I] \qquad \frac{A \wedge B}{B}\ [\wedge E_R] \qquad \frac{A \wedge B}{A}\ [\wedge E_L]$$

1.3 Classical Natural Deduction

It includes all the Constructive Natural Deduction rules, plus:

$$\frac{}{A \vee \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

1.4 Truth Tables

A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

A	B	$A \wedge B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

A	B	$A \rightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

A	$\neg A$
T	**F**
F	**T**

$\top$
T

$\bot$
F

2 Predicate Logic

2.1 Syntax

The syntax of predicate logic is defined by the following grammar:

$$
\begin{array}{lcl}
t & ::= & x \mid f(t, \ldots, t) \\
P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P
\end{array}
$$

There are two special predicate symbols of arity 0: $\top$ which stands for True, and $\bot$ which stands for False.

2.2 Natural Deduction

The Natural Deduction rules for Predicate Logic include all Proposition Logic rules plus the following rules:

$$
\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \qquad \begin{array}{c}\overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q\end{array}}{Q}\ 1\ [\exists E]
$$

Side conditions:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists I]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

2.3 Semantics

Given a signature: $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leq i \leq n$
- of predicate symbols p_i of arity j_i, for $1 \leq i \leq m$

a model M is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n} \rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m} \rangle \rangle$

(a) of a non-empty domain D

(b) interpretations $\mathcal{F}_{f_i} \in D^{k_i} \rightarrow D$ for function symbols f_i

(c) interpretations $\mathcal{R}_{p_i} \subseteq D^{j_i}$ for predicate symbols p_i

A variable valuation v is a partial function from variables to D

Given a model M and a variable valuation v, we assign meaning to terms and formulas as follows:

- Meaning of terms:
  - $[\![x]\!]_v^M = v(x)$
  - $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$
- Meaning of formulas:
  - $\models_{M,v} \top$ is true
  - $\models_{M,v} \bot$ is false
  - $\models_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
  - $\models_{M,v} \neg P$ iff $\neg \models_{M,v} P$
  - $\models_{M,v} P \wedge Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
  - $\models_{M,v} P \vee Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
  - $\models_{M,v} P \rightarrow Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
  - $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x \mapsto d)} P$
  - $\models_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\models_{M,(v,x \mapsto d)} P$

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.