Calculators may be used in this examination provided they are not capable of being used to store alphabetical information other than hexadecimal numbers

UNIVERSITY OF BIRMINGHAM

School of Computer Science

Mathematical and Logical Foundations of Computer Science

Main Summer Examinations 2024

Time allowed: 2 hours

[Answer all questions]

Note

Answer ALL questions. Each question will be marked out of 20. The paper will be marked out of 80, which will be rescaled to a mark out of 100.

Question 1 [Numbers & Sets]

(a) Let `x` be a variable storing a 10-bit floating point encoding with 1 bit for sign, 4 bits for exponent, and 5 bits for mantissa.

(i) Currently, `x` stores the bit-pattern 1 1110 00100.
What number $r \in \mathbb{R}$ does `x` represent? Explain your answer. **[3 marks]**

Let `trunc` be the function from $\mathbb{R}$ to $\mathbb{Z}$ which truncates any real number to an integer. **Meaning:** for $x \in \mathbb{R}$, $\texttt{trunc}(x) \in \mathbb{Z}$ and $0 \leq x - \texttt{trunc}(x) < 1$.

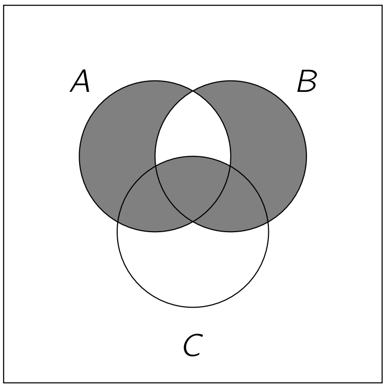
(ii) Is the function `trunc` injective? surjective? bijective? Explain your answers. **[4 marks]**

Let `y` be a variable storing an 8-bit two's complement integer encoding.

(iii) Let $n = \texttt{trunc}(r)$ for $r \in \mathbb{R}$ as obtained in part (i).
How would n be represented into variable `y`? Explain your answer. **[3 marks]**

(b) Let A, B and C be three sets.

(i) Write a set expression, using the union ($\cup$), intersection ($\cap$) and difference ($\setminus$) operators, that describes the set X shaded in the following Venn diagram.



[2 marks]

Now take the three sets A, B and C to be defined as follows:

$$A = \{n \in \mathbb{N} \mid n \equiv 0[2]\}$$
$$B = \{n \in \mathbb{Z} \mid n^2 \leq 10\}$$
$$C = \{x \in \mathbb{R} \mid x \leq -\sqrt{2}\}$$

(ii) Which of the following numbers belong to the set X that you gave in (i)?

$$-18, -3, \sqrt{2}, 24$$

Explain your answer. **[4 marks]**

(iii) Among these three sets, one is finite, one is countably infinite and one is uncountable. Which is which? Explain your answer briefly. **[4 marks]**

Hint: You can use that $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ is uncountable.

Question 2 [Linear Algebra]

Throughout this question, the field is real numbers (with usual addition and multiplication).

(a) State the definition of $\vec{x}$ and α being eigenvector and eigenvalue of a matrix A. **[2 marks]**

(b) Determine the eigenvectors and eigenvalues of the following matrices (show all your working and justification)

$$A_1 = \begin{pmatrix} 3 & 1 \\ 0 & -27 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 4/7 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

[10 marks]

(c) Find the inverse of the following matrix, using Gaussian Elimination (show all your working)

$$A_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1/2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

[5 marks]

(d) Find the determinant of the following matrix (show all your working and justification)

$$A_4 = \begin{pmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \\ -1 & 4 & 1 \end{pmatrix}$$

[3 marks]

Question 3 [Propositional Logic]

Let G be the formula $(\neg\neg A \vee B) \to \neg(\neg B \wedge \neg A)$, and
let H be the formula $(A \wedge \neg B \to C) \to (A \to \neg B) \to \neg C \to \neg A$.
(See the logic appendix below for the Natural Deduction rules for Propositional Logic.)

(a) Provide parse trees corresponding to G and H. **[4 marks]**

(b) Provide a constructive Natural Deduction proof of G. **[6 marks]**

(c) Provide a constructive Natural Deduction proof of H. **[6 marks]**

(d) Provide: (i) an example of a formula that is satisfiable, but not valid, and (ii) an example of a formula that is not satisfiable. Both formulas should use the variables A and B and no others. **[4 marks]**

Question 4 [Predicate Logic]

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\neg\exists x.x < 0$
- let S_2 be $\forall x.\forall y.\neg x < y \to y \leq x$
- let S_3 be $\exists x.\neg 0 < x$

(See the logic appendix below for the Natural Deduction rules for Predicate Logic.)

(a) Provide a constructive Natural Deduction proof of:

$$(S_2) \to (S_3) \to \exists x.x \leq 0$$

[6 marks]

(b) Provide a constructive Natural Deduction proof of:

$$(S_1) \to (S_2) \to \forall x.0 \leq x$$

[8 marks]

(c) Provide a model M_1 such that $\vDash_{M_1} S_2$ and a model M_2 such that $\vDash_{M_2} \neg S_2$. **[6 marks]**

End of Questions – See below for the Logic appendix

Logic appendix

1 Propositional Logic

1.1 Syntax

The syntax of propositional logic formulas is defined by the following grammar, where a ranges over atomic propositions (atoms):

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \to P \mid \neg P$$

There are two special atoms: $\top$ which stands for True, and $\bot$ which stands for False. We use the following precedence and associativity rules:

- Precedence: in decreasing order of precedence $\neg, \wedge, \vee, \to$.
- Associativity: all operators are right associative

1.2 Constructive Natural Deduction

$$\frac{\bot}{A}\ [\bot E] \qquad \frac{}{\top}\ [\top I] \qquad \frac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ B\end{array}}{A \to B}\ 1\ [\to I] \qquad \frac{A \to B \quad A}{B}\ [\to E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ \bot\end{array}}{\neg A}\ 1\ [\neg I] \qquad \frac{\neg A \quad A}{\bot}\ [\neg E]$$

$$\frac{A}{A \vee B}\ [\vee I_L] \qquad \frac{A}{B \vee A}\ [\vee I_R] \qquad \frac{A \vee B \quad A \to C \quad B \to C}{C}\ [\vee E]$$

$$\frac{A \quad B}{A \wedge B}\ [\wedge I] \qquad \frac{A \wedge B}{B}\ [\wedge E_R] \qquad \frac{A \wedge B}{A}\ [\wedge E_L]$$

1.3 Classical Natural Deduction

It includes all the Constructive Natural Deduction rules, plus:

$$\frac{}{A \vee \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

1.4 Truth Tables

A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

A	B	$A \wedge B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

A	B	$A \rightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

A	$\neg A$
T	**F**
F	**T**

$\top$
T

$\bot$
F

2 Predicate Logic

2.1 Syntax

The syntax of predicate logic is defined by the following grammar:

$$\begin{array}{lll} t & ::= & x \mid f(t, \ldots, t) \\ P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P \end{array}$$

There are two special predicate symbols of arity 0: $\top$ which stands for True, and $\bot$ which stands for False.

2.2 Natural Deduction

The Natural Deduction rules for Predicate Logic include all Proposition Logic rules plus the following rules:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \qquad \begin{array}{c} \overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q \end{array}}{Q}\ 1\ [\exists E]$$

Side conditions:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists I]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

2.3 Semantics

Given a signature: $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leq i \leq n$
- of predicate symbols p_i of arity j_i, for $1 \leq i \leq m$

a model M is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n} \rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m} \rangle \rangle$

(a) of a non-empty domain D

(b) interpretations $\mathcal{F}_{f_i} \in D^{k_i} \to D$ for function symbols f_i

(c) interpretations $\mathcal{R}_{p_i} \subseteq D^{j_i}$ for function symbols p_i

A variable valuation v is a partial function from variables to D

Given a model M and a variable valuation v, we assign meaning to terms and formulas as follows:

- Meaning of terms:
  - $[\![x]\!]_v^M = v(x)$
  - $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$
- Meaning of formulas:
  - $\vDash_{M,v} \top$ is true
  - $\vDash_{M,v} \bot$ is false
  - $\vDash_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
  - $\vDash_{M,v} \neg P$ iff $\neg \vDash_{M,v} P$
  - $\vDash_{M,v} P \wedge Q$ iff $\vDash_{M,v} P$ and $\vDash_{M,v} Q$
  - $\vDash_{M,v} P \vee Q$ iff $\vDash_{M,v} P$ or $\vDash_{M,v} Q$
  - $\vDash_{M,v} P \to Q$ iff $\vDash_{M,v} Q$ whenever $\vDash_{M,v} P$
  - $\vDash_{M,v} \forall x.P$ iff for every $d \in D$ we have $\vDash_{M,(v,x \mapsto d)} P$
  - $\vDash_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\vDash_{M,(v,x \mapsto d)} P$

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

Any students found with non-permitted items upon their person during the examination, or who fail to comply with Examination rules may be subject to Student Conduct procedures.