LC Mathematical and Logical Foundations of Computer Science Solutions

Main Summer Examinations 2023

This is last year's exam. Since the content is slightly different we've suggested slightly different questions in red.
Some of the content will not be covered or will be covered later (linear algebra and predicate logic), as indicated below.

Non-alpha only

Note

Answer ALL questions. Each question will be marked out of 20. The paper will be marked out of 80, which will be rescaled to a mark out of 100.

Question 1 [Numbers & Sets]

Usual notation for sets may be used, including the following.

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \{(x, y) \mid x \in A, y \in B\} \\ A + B &\stackrel{\text{def}}{=} \{(0, x) \mid x \in A\} \cup \{(1, y) \mid y \in B\} \\ A^* &\stackrel{\text{def}}{=} \text{the set of all lists of elements of } A \\ [a\,..\,b) &\stackrel{\text{def}}{=} \{n \in \mathbb{Z} \mid a \leqslant n < b\} \\ [a\,..\,b] &\stackrel{\text{def}}{=} \{n \in \mathbb{Z} \mid a \leqslant n \leqslant b\} \end{aligned}$$

Try this instead:

16 bits: 1 sign, 5 exponent, 10 mantissa

(a) In my program, `x` and `y` are variables, each storing an 8+8-bit representation of a *positive* real number. (This means 8 bits for the fractional part of the mantissa, and 8 bits for the exponent using a bias of 2^7.)

(i) Currently, `x` stores the bit-pattern 1001 1000 1000 0010. What number does this represent? Explain your answer. **[4 marks]**

sign, mantissa, exponent

(ii) Although the number stored in `y` is greater than 0, is it possible that executing the instruction `x = x + y` can leave `x` unchanged? Explain your answer. **[4 marks]**

(b) My online shop has two kinds of customer: regular and premium. A regular customer has just an ID number and a name. A premium customer has an ID number, a name, and the name of their customer service contact. (An ID number is any natural number.) Give a set whose elements are all possible customer records. You can take `Char` to be the set of all characters that can appear in a string. **[4 marks]**

use String in your answer which is defined as char*

(c) Let C be the set $\{4n + 3 \mid n \in \mathbb{N}\}$. Let $f : C \to [0\,..\,5)$ be the function sending x to x mod 5.

(i) Is this function injective? Explain your answer. **[4 marks]**

(ii) What are the elements of $C/\ker(f)$? Recall that $\ker(f)$ is the equivalence relation on C that relates any two elements $x, y \in C$ such that $f(x) = f(y)$. **[4 marks]**

not covered this year

Model answer / LOs / Creativity:

(a) (i) It represents $a \times 2^n$, where

$$\begin{aligned} a &= 1.10011000_2 \\ &= 1 + \frac{1}{2} + \frac{1}{16} + \frac{1}{32} \\ &= 1\frac{19}{32} \\ n &= 10000010_2 - 2^7 \\ &= 4 \end{aligned}$$

So it represents $1\frac{19}{32} \times 2^4$. This can be simplified to $25\frac{1}{2}$ (not required).

(ii) Yes, it's possible. For example, suppose that `y` stores the bit-pattern 0000 0000 0000 0000, which represents 2^{-128}. Adding this to `x` gives $25\frac{1}{2} + 2^{-128}$, which rounds to $25\frac{1}{2}$.

(b) The set of strings is $\texttt{Char}^*$, so the set of records is

$$\mathbb{N} \times \texttt{Char}^* + \mathbb{N} \times \texttt{Char}^* \times \texttt{Char}^*$$

(c) (i) $f(3) = 3$ and $f(23) = 3$, but $3 \neq 23$, so f isn't injective.

(ii) The elements of $A/\mathsf{ker}(f)$ are as follows:

$$\{20n + 3 \mid n \in \mathbb{N}\}, \{20n + 7 \mid n \in \mathbb{N}\}, \{20n + 11 \mid n \in \mathbb{N}\},$$
$$\{20n + 15 \mid n \in \mathbb{N}\}, \{20n + 19 \mid n \in \mathbb{N}\}$$

Learning outcomes:

- "Solve mathematical problems in algebra and set theory."
- "Apply mathematical techniques to solve a problem within a computer science setting".

Creativity is in parts (b) and (c).

Not yet covered

Question 2 [Linear Algebra]

Throughout this question, the field is rational numbers (with usual addition and multiplication).

(a) Consider the vector space of four-tuples of rational numbers. Show that the following four vectors **are not** linearly independent:

$$\vec{a_1} = \begin{pmatrix} 1 \\ 11 \\ 13 \\ 12 \end{pmatrix} \quad \vec{a_2} = \begin{pmatrix} 2 \\ 6 \\ 10 \\ 6 \end{pmatrix} \quad \vec{a_3} = \begin{pmatrix} 4 \\ 7 \\ 3 \\ 4 \end{pmatrix} \quad \vec{a_4} = \begin{pmatrix} 3 \\ 1 \\ 7 \\ 0 \end{pmatrix}$$

[7 marks]

(b) Consider the vector space of three-tuples of rational numbers. Show that the following three vectors form an orthogonal basis:

$$\vec{x_1} = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} \quad \vec{x_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{x_3} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

[10 marks]

(c) Does the vector space from part (b) have a basis which has four vectors? If yes, then give an example of such a basis. If no, explain why this is not possible.

[3 marks]

Model answer / LOs / Creativity:

(a) This can be solved by Gaussian Elimination, but a creative solution is to observe that $\vec{a_1} + \vec{a_4} = 2 \cdot \vec{a_2}$.

Marking scheme: 0 marks if not attempted. 7 marks if fully correct. Partial credit (3-4) marks if they have the rough idea but make a mistake somewhere, maybe an arithmetic mistake in Gaussian elimination.

(b) First we observe that the three vectors are linearly independent: suppose there exist rational numbers a_1, a_2, a_3 such that $\sum_{i=1}^{3} a_i \cdot \vec{x_i} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Expanding the left-hand size of the equation, we get that $\sum_{i=1}^{3} a_i \cdot \vec{x_i} = \begin{pmatrix} 2a_3 \\ -5a_1 \\ a_2 \end{pmatrix}$, i.e., $a_1 = a_2 = a_3 = 0$. Hence, the three given vectors are linearly independent.

Next, we show that any three-tuple, say $\overrightarrow{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ of rational numbers can be expressed as a linear combination of $\overrightarrow{x_1}, \overrightarrow{x_2}$ and $\overrightarrow{x_3}$. This follows since we have $\frac{y_2}{-5} \cdot \overrightarrow{x_1} + y_3 \cdot \overrightarrow{x_2} + \frac{y_1}{2} \cdot \overrightarrow{x_3} = \frac{y_2}{-5} \cdot \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} + y_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{y_1}{2} \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \overrightarrow{y}$.

Finally, for orthogonality we calculate each of the three pairwise inner products and observe that they are the zero vector.

Marking scheme: 3 marks for showing linear independence. 4 marks for showing every element of the vector space can be expressed using the given three vectors. 3 marks (1 for each of the three pairwise inner products) for showing basis is orthogonal

(c) No, this is not possible. We have constructed a basis with three vectors in part (b). In the lectures, we have seen that every basis of a vector space has the same number of vectors hence a basis with 4 vectors is not possible.

Marking scheme: 0 marks if not attempted. 3 marks even if just stated (without proof) the fact covered in the lectures that every basis has the same number of vectors and we have constructed a basis with three vectors in part (b)

Learning outcomes:

- "Solve mathematical problems in algebra and set theory." (a, b, c);
- "Apply mathematical techniques to solve a problem within a computer science setting" (a, b, c).

Creativity is not necessary, but can help simplify proof of (a) quite a bit. For (b), observing right away that the given vectors are essentially the standard basis (mentioned in the lectures) can also shorten the arguments.

Question 3 [Propositional Logic]

Let F be the formula $(A \vee \neg B) \to B \to A$, and
let G be the formula $(A \to B) \to \neg B \to \neg(A \vee B)$.
(See the logic cheat sheet below for the Natural Deduction rules for Propositional Logic.)

(a) Provide parse trees corresponding to F and G. **[4 marks]**

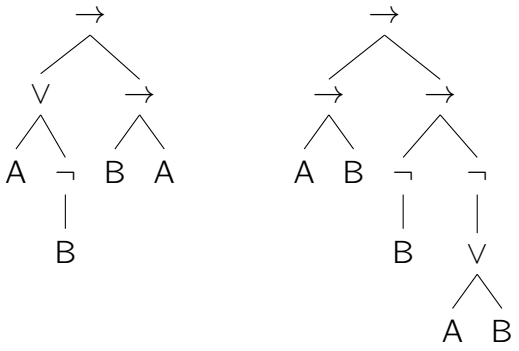
(b) Provide a constructive Natural Deduction proof of F. **[6 marks]**

(c) Provide a constructive Natural Deduction proof of G. **[6 marks]**

(d) Is F satisfiable? Is G valid? Justify your answers. **[4 marks]**

Model answer / LOs / Creativity:

(a) The parse trees of F and G are



Marking scheme: 2 marks for each tree; 1 mark if partially correct.

(b) Here is a proof of F:

$$\dfrac{\dfrac{\overline{A \vee \neg B}^{\,1} \quad \dfrac{\overline{A}^{\,3}}{A \to A}\,3\,[\to I] \quad \dfrac{\dfrac{\dfrac{\overline{B}^{\,2} \quad \overline{\neg B}^{\,4}}{\bot}\,[\neg E]}{A}\,[\bot\, E]}{\neg B \to A}\,4\,[\to I]}{A}\,[\vee E]}{B \to A}\,2\,[\to I]}{(A \vee \neg B) \to B \to A}\,1\,[\to I]$$

Marking scheme: 1 if attempted; 2 if introduced hypotheses; 3 if realized the $\vee$ has to be eliminated; 4 if just a few mistakes; 5 if near perfect (a few missing labels); 6 if perfect.

(c) Here is a proof of G:

$$\dfrac{\dfrac{\overline{A \vee B}^{\,3} \quad \dfrac{\dfrac{\overline{A \to B}^{\,1} \quad \overline{A}^{\,4}}{B}\,[\to E] \quad \overline{\neg B}^{\,2}}{\bot}\,[\neg E] \quad \dfrac{\dfrac{\overline{B}^{\,5} \quad \overline{\neg B}^{\,2}}{\bot}\,[\neg E]}{B \to \bot}\,5\,[\to I]}{\bot}\,[\vee E]}{\neg(A \vee B)}\,3\,[\neg I]}{\neg B \to \neg(A \vee B)}\,2\,[\to I]}{(A \to B) \to \neg B \to \neg(A \vee B)}\,1\,[\to I]$$

Marking scheme: 1 if attempted; 2 if introduced hypotheses; 3 if realized the $\vee$ has to be eliminated; 4 if just a few mistakes; 5 if near perfect (a few missing labels); 6 if perfect.

(d) Both formulas are provable so by soundness are valid, and therefore satisfiable too. This question can be answered by drawing a truth table or by providing one valuation that make the formula true in the case of satisfiability.

Marking scheme: 2 marks for each question; 1 mark if not fully justified

Learning outcomes:

- "Understand and apply algorithms for key problems in logic such as satisfiability." (b, c, & d);
- "Write formal proofs for propositional and predicate logic" (b & c);
- "Apply logical techniques to solve a problem within a computer science setting" (a, b, c, & d).

Creativity will be required to come up with proofs.

Not yet covered

Non-alpha only

Question 4 [Predicate Logic]

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\neg\exists x.0 \leq x$
- let S_2 be $\forall x.\forall y.x < y \rightarrow x \leq \mathtt{succ}(y)$

(See the logic cheat sheet below for the Natural Deduction rules for Predicate Logic.)

(a) Provide a constructive Natural Deduction proof of:

$$(S_1) \rightarrow \forall x.\neg 0 \leq x$$

[6 marks]

(b) Provide a constructive Natural Deduction proof of:

$$(S_1) \rightarrow (S_2) \rightarrow \neg\exists x.0 < x$$

[8 marks]

(c) Provide a model M_1 such that $\vDash_{M_1} \exists x.\exists y.x \leq y \wedge \neg x < y$ and a model M_2 such that $\vDash_{M_2} \neg(\exists x.\exists y.x \leq y \wedge \neg x < y)$. **[6 marks]**

Model answer / LOs / Creativity:

(a) Here is a proof of the formula:

$$\dfrac{\dfrac{\dfrac{\overline{S_1}^{\,1} \quad \dfrac{\overline{0 \leq x}^{\,2}}{\exists x.0 \leq x}\,[\exists I]}{\bot}\,[\neg E]}{\neg 0 \leq x}\,2\,[\neg I]}{\forall x.\neg 0 \leq x}\,[\forall I]}{S_1 \rightarrow \forall x.\neg 0 \leq x}\,1\,[\rightarrow I]$$

Marking scheme: 1 if attempted; 2 if realized the $\forall$ must be introduced; 3 if found the contradiction to prove; 4 if instantiated the existential correctly; 5 if near perfect (e.g., missing labels); 6 if perfect.

(b) Here is a proof of the formula:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{S_2}\,2}{\forall y.0 < y \to 0 \leq \mathtt{succ}(y)}\,[\forall E]}{0 < x \to 0 \leq \mathtt{succ}(x)}\,[\forall E] \quad \dfrac{}{0 < x}\,4}{0 \leq \mathtt{succ}(x)}\,[\to E]}{\exists x.0 \leq x}\,[\exists I] \quad \dfrac{}{S_1}\,1}{\bot}\,[\neg E]}{\exists x.0 < x}\,3 \quad \bot}\,4\,[\exists E]}{\neg\exists x.0 < x}\,3\,[\neg I]}{S_2 \to \neg\exists x.0 < x}\,2\,[\to I]}{S_1 \to S_2 \to \neg\exists x.0 < x}\,1\,[\to I]$$

Marking scheme: 1 if attempted; 2 if introduced hypotheses; 3 if realized the $\exists$ has to be eliminated; 4 if the elimination of the $\exists$ was done correctly; 5 if found the contradiction to prove; 6 if instantiated the existential correctly; 7 if near perfect (a few missing labels); 8 if perfect.

(c) For example, the models

- $M_1 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n+1 \rangle, \langle \{\langle n, m \rangle \mid n < m\}, \{\langle n, m \rangle \mid n \leq m\} \rangle \rangle$
- and $M_1' = \langle \{0\}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \emptyset, \{\langle n, m \rangle \mid \text{True}\} \rangle \rangle$

are models of $\exists x.\exists y.x \leq y \wedge \neg x < y$;

The models

- $M_2 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n+1 \rangle, \langle \{\langle n, m \rangle \mid n \leq m\}, \{\langle n, m \rangle \mid n < m\} \rangle \rangle$
- and $M_2' = \langle \{0\}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid \text{True}\}, \emptyset \rangle \rangle$

are models of $\neg\exists x.\exists y.x \leq y \wedge \neg x < y$.

Marking scheme: 3 marks for each question; 1 or 2 marks if the functions/sets are not quite well-formed.

Learning outcomes:

- "Understand and apply algorithms for key problems in logic such as satisfiability." (a, b, & c);
- "Write formal proofs for propositional and predicate logic" (a & b);
- "Apply logical techniques to solve a problem within a computer science setting" (a, b, & c).

Creativity will be required to come up with proofs and models.

End of Questions – See below for the Logic cheat sheet

Logic cheat sheet

1 Propositional Logic

1.1 Syntax

The syntax of propositional logic formulas is defined by the following grammar, where a ranges over atomic propositions (atoms):

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \to P \mid \neg P$$

There are two special atoms: $\top$ which stands for True, and $\bot$ which stands for False. We use the following precedence and associativity rules:

- Precedence: in decreasing order of precedence $\neg, \wedge, \vee, \to$.
- Associativity: all operators are right associative

1.2 Constructive Natural Deduction

$$\frac{\bot}{A}\ [\bot E] \qquad \frac{}{\top}\ [\top I] \qquad \frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ B\end{array}}{A \to B}\ 1\ [\to I] \qquad \frac{A \to B \quad A}{B}\ [\to E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ \bot\end{array}}{\neg A}\ 1\ [\neg I] \qquad \frac{\neg A \quad A}{\bot}\ [\neg E]$$

$$\frac{A}{A \vee B}\ [\vee I_L] \qquad \frac{A}{B \vee A}\ [\vee I_R] \qquad \frac{A \vee B \quad A \to C \quad B \to C}{C}\ [\vee E]$$

$$\frac{A \quad B}{A \wedge B}\ [\wedge I] \qquad \frac{A \wedge B}{B}\ [\wedge E_R] \qquad \frac{A \wedge B}{A}\ [\wedge E_L]$$

1.3 Classical Natural Deduction

It includes all the Constructive Natural Deduction rules, plus:

$$\frac{}{A \vee \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

1.4 Truth Tables

A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

A	B	$A \wedge B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

A	B	$A \to B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

A	$\neg A$
T	**F**
F	**T**

$\top$
T

$\bot$
F

2 Predicate Logic

2.1 Syntax

The syntax of predicate logic is defined by the following grammar:

$$\begin{array}{lcl} t & ::= & x \mid f(t, \ldots, t) \\ P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \to P \mid \forall x.P \mid \exists x.P \end{array}$$

There are two special predicate symbols of arity 0: $\top$ which stands for True, and $\bot$ which stands for False.

2.2 Natural Deduction

The Natural Deduction rules for Predicate Logic include all Proposition Logic rules plus the following rules:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \qquad \begin{array}{c}\overline{P[x\backslash y]}^{\,1}\\ \vdots \\ Q\end{array}}{Q}\ 1\ [\exists E]$$

Side conditions:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists I]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

2.3 Semantics

Given a signature: $\langle \langle f_1^{k_1}, \ldots, f_n^{k_n} \rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m} \rangle \rangle$

- of function symbols f_i of arity k_i, for $1 \leq i \leq n$
- of predicate symbols p_i of arity j_i, for $1 \leq i \leq m$

a model M is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n} \rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m} \rangle \rangle$

(a) of a non-empty domain D

(b) interpretations $\mathcal{F}_{f_i} \in D^{k_i} \to D$ for function symbols f_i

(c) interpretations $\mathcal{R}_{p_i} \subseteq D^{j_i}$ for function symbols p_i

A variable valuation v is a partial function from variables to D

Given a model M and a variable valuation v, we assign meaning to terms and formulas as follows:

- Meaning of terms:
  - $[\![x]\!]_v^M = v(x)$
  - $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$
- Meaning of formulas:
  - $\vDash_{M,v} \top$ is true
  - $\vDash_{M,v} \bot$ is false
  - $\vDash_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
  - $\vDash_{M,v} \neg P$ iff $\neg \vDash_{M,v} P$
  - $\vDash_{M,v} P \wedge Q$ iff $\vDash_{M,v} P$ and $\vDash_{M,v} Q$
  - $\vDash_{M,v} P \vee Q$ iff $\vDash_{M,v} P$ or $\vDash_{M,v} Q$
  - $\vDash_{M,v} P \to Q$ iff $\vDash_{M,v} Q$ whenever $\vDash_{M,v} P$
  - $\vDash_{M,v} \forall x.P$ iff for every $d \in D$ we have $\vDash_{M,(v,x \mapsto d)} P$
  - $\vDash_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\vDash_{M,(v,x \mapsto d)} P$