Calculators may be used in this examination provided they are not capable of being used to store alphabetical information other than hexadecimal numbers

UNIVERSITY OF BIRMINGHAM

School of Computer Science

LC Mathematical and Logical Foundations of Computer Science

Main Summer Examinations 2023

Time allowed: 2 hours

[Answer all questions]

Note

Answer ALL questions. Each question will be marked out of 20. The paper will be marked out of 80, which will be rescaled to a mark out of 100.

Question 1 [Numbers & Sets]

Usual notation for sets may be used, including the following.

$$
\begin{aligned}
A \times B &\stackrel{\text{def}}{=} \{(x, y) \mid x \in A, y \in B\} \\
A + B &\stackrel{\text{def}}{=} \{(0, x) \mid x \in A\} \cup \{(1, y) \mid y \in B\} \\
A^* &\stackrel{\text{def}}{=} \text{the set of all lists of elements of } A \\
[a\mathinner{..}b) &\stackrel{\text{def}}{=} \{n \in \mathbb{Z} \mid a \leqslant n < b\} \\
[a\mathinner{..}b] &\stackrel{\text{def}}{=} \{n \in \mathbb{Z} \mid a \leqslant n \leqslant b\}
\end{aligned}
$$

(a) In my program, `x` and `y` are variables, each storing an 8+8-bit representation of a *positive* real number. (This means 8 bits for the fractional part of the mantissa, and 8 bits for the exponent using a bias of 2^7.)

(i) Currently, `x` stores the bit-pattern 1001 1000 1000 0010. What number does this represent? Explain your answer. **[4 marks]**

(ii) Although the number stored in `y` is greater than 0, is it possible that executing the instruction `x = x + y` can leave `x` unchanged? Explain your answer. **[4 marks]**

(b) My online shop has two kinds of customer: regular and premium. A regular customer has just an ID number and a name. A premium customer has an ID number, a name, and the name of their customer service contact. (An ID number is any natural number.) Give a set whose elements are all possible customer records. You can take `Char` to be the set of all characters that can appear in a string. **[4 marks]**

(c) Let C be the set $\{4n + 3 \mid n \in \mathbb{N}\}$. Let $f : C \to [0\mathinner{..}5)$ be the function sending x to $x \bmod 5$.

(i) Is this function injective? Explain your answer. **[4 marks]**

(ii) What are the elements of $C/\ker(f)$? Recall that $\ker(f)$ is the equivalence relation on C that relates any two elements $x, y \in C$ such that $f(x) = f(y)$. **[4 marks]**

Question 2 [Linear Algebra]

Throughout this question, the field is rational numbers (with usual addition and multiplication).

(a) Consider the vector space of four-tuples of rational numbers. Show that the following four vectors **are not** linearly independent:

$$\vec{a_1} = \begin{pmatrix} 1 \\ 11 \\ 13 \\ 12 \end{pmatrix} \qquad \vec{a_2} = \begin{pmatrix} 2 \\ 6 \\ 10 \\ 6 \end{pmatrix} \qquad \vec{a_3} = \begin{pmatrix} 4 \\ 7 \\ 3 \\ 4 \end{pmatrix} \qquad \vec{a_4} = \begin{pmatrix} 3 \\ 1 \\ 7 \\ 0 \end{pmatrix}$$

[7 marks]

(b) Consider the vector space of three-tuples of rational numbers. Show that the following three vectors form an orthogonal basis:

$$\vec{x_1} = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} \qquad \vec{x_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{x_3} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

[10 marks]

(c) Does the vector space from part (b) have a basis which has four vectors? If yes, then give an example of such a basis. If no, explain why this is not possible.

[3 marks]

Question 3 [Propositional Logic]

Let F be the formula $(A \vee \neg B) \rightarrow B \rightarrow A$, and
let G be the formula $(A \rightarrow B) \rightarrow \neg B \rightarrow \neg(A \vee B)$.
(See the logic cheat sheet below for the Natural Deduction rules for Propositional Logic.)

(a) Provide parse trees corresponding to F and G. **[4 marks]**

(b) Provide a constructive Natural Deduction proof of F. **[6 marks]**

(c) Provide a constructive Natural Deduction proof of G. **[6 marks]**

(d) Is F satisfiable? Is G valid? Justify your answers. **[4 marks]**

Question 4 [Predicate Logic]

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\neg\exists x.0 \leq x$
- let S_2 be $\forall x.\forall y.x < y \rightarrow x \leq \texttt{succ}(y)$

(See the logic cheat sheet below for the Natural Deduction rules for Predicate Logic.)

(a) Provide a constructive Natural Deduction proof of:

$$(S_1) \rightarrow \forall x.\neg 0 \leq x$$

[6 marks]

(b) Provide a constructive Natural Deduction proof of:

$$(S_1) \rightarrow (S_2) \rightarrow \neg\exists x.0 < x$$

[8 marks]

(c) Provide a model M_1 such that $\models_{M_1} \exists x.\exists y.x \leq y \wedge \neg x < y$ and a model M_2 such that $\models_{M_2} \neg(\exists x.\exists y.x \leq y \wedge \neg x < y)$, **[6 marks]**

End of Questions – See below for the Logic cheat sheet

Logic cheat sheet

1 Propositional Logic

1.1 Syntax

The syntax of propositional logic formulas is defined by the following grammar, where a ranges over atomic propositions (atoms):

$$P ::= a \mid P \land P \mid P \lor P \mid P \to P \mid \neg P$$

There are two special atoms: $\top$ which stands for True, and $\bot$ which stands for False. We use the following precedence and associativity rules:

- Precedence: in decreasing order of precedence $\neg, \land, \lor, \to$.
- Associativity: all operators are right associative

1.2 Constructive Natural Deduction

$$\frac{\bot}{A}\ [\bot E] \qquad \frac{}{\top}\ [\top I] \qquad \frac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ B\end{array}}{A \to B}\ 1\ [\to I] \qquad \frac{A \to B \quad A}{B}\ [\to E]$$

$$\frac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ \bot\end{array}}{\neg A}\ 1\ [\neg I] \qquad \frac{\neg A \quad A}{\bot}\ [\neg E]$$

$$\frac{A}{A \lor B}\ [\lor I_L] \qquad \frac{A}{B \lor A}\ [\lor I_R] \qquad \frac{A \lor B \quad A \to C \quad B \to C}{C}\ [\lor E]$$

$$\frac{A \quad B}{A \land B}\ [\land I] \qquad \frac{A \land B}{B}\ [\land E_R] \qquad \frac{A \land B}{A}\ [\land E_L]$$

1.3 Classical Natural Deduction

It includes all the Constructive Natural Deduction rules, plus:

$$\frac{}{A \lor \neg A}\ [LEM] \qquad \frac{\neg\neg A}{A}\ [DNE]$$

1.4 Truth Tables

A	B	$A \vee B$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

A	B	$A \wedge B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

A	B	$A \rightarrow B$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

A	$\neg A$
T	**F**
F	**T**

$\top$
T

$\bot$
F

2 Predicate Logic

2.1 Syntax

The syntax of predicate logic is defined by the following grammar:

$$\begin{array}{lcl} t & ::= & x \mid f(t, \ldots, t) \\ P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P \end{array}$$

There are two special predicate symbols of arity 0: $\top$ which stands for True, and $\bot$ which stands for False.

2.2 Natural Deduction

The Natural Deduction rules for Predicate Logic include all Proposition Logic rules plus the following rules:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \quad \begin{array}{c} \overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q \end{array}}{Q}\ 1\ [\exists E]$$

Side conditions:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists I]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

2.3 Semantics

Given a signature: $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leq i \leq n$
- of predicate symbols p_i of arity j_i, for $1 \leq i \leq m$

a model M is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n} \rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m} \rangle \rangle$

(a) of a non-empty domain D

(b) interpretations $\mathcal{F}_{f_i} \in D^{k_i} \to D$ for function symbols f_i

(c) interpretations $\mathcal{R}_{p_i} \subseteq D^{j_i}$ for function symbols p_i

A variable valuation v is a partial function from variables to D

Given a model M and a variable valuation v, we assign meaning to terms and formulas as follows:

- Meaning of terms:
  - $[\![x]\!]_v^M = v(x)$
  - $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$
- Meaning of formulas:
  - $\models_{M,v} \top$ is true
  - $\models_{M,v} \bot$ is false
  - $\models_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
  - $\models_{M,v} \neg P$ iff $\neg \models_{M,v} P$
  - $\models_{M,v} P \wedge Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
  - $\models_{M,v} P \vee Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
  - $\models_{M,v} P \to Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
  - $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x\mapsto d)} P$
  - $\models_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\models_{M,(v,x\mapsto d)} P$

Do not complete the attendance slip, fill in the front of the answer book or turn over the question paper until you are told to do so

Important Reminders

- Coats/outwear should be placed in the designated area.
- Unauthorised materials (e.g. notes or Tippex) must be placed in the designated area.
- Check that you do not have any unauthorised materials with you (e.g. in your pockets, pencil case).
- Mobile phones and smart watches **must** be switched off and placed in the designated area or under your desk. They must not be left on your person or in your pockets.
- You are not permitted to use a mobile phone as a clock. If you have difficulty seeing a clock, please alert an Invigilator.
- You are not permitted to have writing on your hand, arm or other body part.
- Check that you do not have writing on your hand, arm or other body part – if you do, you must inform an Invigilator immediately
- Alert an Invigilator immediately if you find any unauthorised item upon you during the examination.

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