UNIVERSITY OF BIRMINGHAM

School of Computer Science

First Year Undergraduate

06-35324

35324 LC Mathematical and Logical Foundations of Computer Science

Main Summer Examinations 2022

[Answer all questions]

35324 LC Mathematical and Logical Foundations of Computer Science

Question 1

(a) **Numbers and sets**

(i) Compute 4^2 and 4^3 modulo 3. From these results, conjecture a property $P(n)$ of 4^n modulo 3. Prove by induction that $P(n)$ is satisfied for all $n \in \mathbb{N}$. **[3 marks]**

(ii) Let $sqrt : \{x \in \mathbb{N} \mid x \geq 0\} \to \{y \in \mathbb{R} \mid y \geq 0\}$ be the square root function from non-negative integers to non-negative reals, i.e. $sqrt(x) = y$ if and only if $x = y^2$.

Is $sqrt$ injective? surjective? bijective? Justify your answers. **[3 marks]**

(iii) Let `float javaSqrt(int x)` be a Java implementation of this square root function (assume it throws an exception if `x` is negative). Discuss the differences between $sqrt$ and `javaSqrt` when considered as mathematical functions between sets. **[4 marks]**

(b) **Linear algebra**
Consider the three points

$$P_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \qquad P_2 = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \qquad P_3 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

(i) Show that these form the corners of an equilateral triangle (i.e. a triangle where all sides have the same length). **[2 marks]**

(ii) The triangle defines a plane M in $\mathbb{R}^3$. Give its parametric representation and its normal form. **[4 marks]**

(iii) Another plane N is given by

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + q \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Find the line of intersection of M and N. **[4 marks]**

Show your working for each part.

Question 2

(a) **Propositional Logic**

Let F be the formula $(A \vee B) \to (B \vee A)$, and
let G be the formula $A \to (B \to (\neg B \vee \neg A)) \to \neg B$.

(i) Provide a constructive Sequent Calculus proof of F. **[4 marks]**

(ii) Provide a constructive Natural Deduction proof of G. **[6 marks]**

(iii) Is G satisfiable? Justify your answer. **[2 marks]**

(b) **Predicate Logic**

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. Consider the following formula that captures a property of the above symbols:

- let S_1 be $\forall x.\neg(x < 0)$

For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc.

(i) Provide a constructive Natural Deduction proof of:

$$S_1 \rightarrow \neg\exists x.\forall y.x < y$$

[4 marks]

(ii) Provide a model M_1 such that $\models_{M_1} \neg(\forall x.\forall y.x < y \rightarrow x \leq y)$, and a model M_2 such that $\models_{M_2} \forall x.\forall y.x \leq y \rightarrow x < y$ **[4 marks]**