UNIVERSITY OF BIRMINGHAM

School of Computer Science

First Year Undergraduate

06-35324

35324 LC Mathematical and Logical Foundations of Computer Science

Resit Examinations 2022

[Answer all questions]

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Question 1

(a) **Numbers and sets**

(i) Define a relation R on pairs $(x, y) \in \mathbb{R}^2$ by

$$((x, y), (x', y')) \in R \quad \text{if and only if} \quad y = y'$$

Show that R is an equivalence relation.
Write down the equivalence class under R of $(1, 2)$. **[4 marks]**

(ii) Consider the following relations:

- $S = \{((x, y), z) \in (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \mid z \text{ is the average of } x \text{ and } y\}$
- $T = \{((x, y), z) \in (\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q} \mid z \text{ is the average of } x \text{ and } y\}$

Are they functions? Justify your answers. **[2 marks]**

(iii) Consider the following piece of pseudo-code for computing the average of a data set, where `data` is an array of numbers of length `max`:

```
total <- 0
i <- 0
while (i < max) {
    total <- total + data[i]
    i <- i + 1 }
return (total / max)
```

Discuss the differences between the output of the pseudo-code and the mathematical average of the data points that might occur if `data` is a array of `int` and `total` is an `int`. Then do the same for the case where `data` is an array of `float` and total is a `float`. (You may assume the implementation is in Java. Consider in particular the case of large data sets.) **[4 marks]**

(b) **Linear algebra**
Consider the three points

$$P_1 = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} \qquad P_3 = \begin{pmatrix} 4 \\ -6 \\ -6 \end{pmatrix}$$

(i) Show that three points are on a line in $\mathbb{R}^3$. We will call this line L. Give a parametric representation of L. **[4 marks]**

(ii) A plane M in $\mathbb{R}^3$ is given by

$$\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

Find the intersection between M and L. **[3 marks]**

(iii) Consider the point

$$Q = \begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix}$$

Is Q on the same side of M as P_1? Justify your answer. **[3 marks]**

Show your working for each part.

Question 2

(a) **Propositional Logic**

Let F be the formula $(A \wedge B) \to (\neg A \vee \neg\neg B)$, and
let G be the formula $(\neg\neg B \to C) \to \neg C \to \neg B$

(i) Provide a constructive Sequent Calculus proof of F. **[4 marks]**

(ii) Provide a constructive Natural Deduction proof of G. **[4 marks]**

(iii) Is G falsifiable? Justify your answer **[2 marks]**

(b) **Predicate Logic**

Consider the following signature:

- Function symbols: $\texttt{zero}$ (arity 0); $\texttt{succ}$ (arity 1)
- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\forall x. x < \texttt{succ}(x)$
- let S_2 be $\forall x. \forall y. x \leq y \to \neg(y < x)$

where for simplicity we write 0 for $\texttt{zero}$, 1 for $\texttt{succ}(\texttt{zero})$, 2 for $\texttt{succ}(\texttt{succ}(\texttt{zero}))$, etc.

(i) Provide a constructive Sequent Calculus proof of:

$$S_1, S_2 \vdash \forall x. \neg(\texttt{succ}(x) \leq x)$$

[6 marks]

(ii) Provide a model M_1 such that $\vDash_{M_1} \neg\forall x. \neg(\texttt{succ}(x) \leq x)$, and a model M_2 such that $\vDash_{M_2} \forall x. \neg(\texttt{succ}(x) \leq x)$ **[4 marks]**