UNIVERSITY OF BIRMINGHAM

School of Computer Science

Mathematical and Logical Foundations of Computer Science Second Class Test

Second Class Test 2020/21

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Question 1 [Linear Algebra]

(a) Consider the following two vectors in $\mathbb{R}^3$:

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

(i) Show that $\vec{v}$ and $\vec{w}$ are linearly independent of each other. **[2 marks]**

(ii) Find a third vector $\vec{u}$ so that $\{\vec{v}, \vec{w}, \vec{u}\}$ form a basis of $\mathbb{R}^3$. **[2 marks]**

(b) The points

$$P_1 = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad P_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

are the corners of a triangle in $\mathbb{R}^3$.

(i) Show that the triangle has a right angle, and say at which corner it occurs. **[4 marks]**

(ii) The triangle defines a plane E in $\mathbb{R}^3$. Give its parametric representation and its normal form. **[4 marks]**

(iii) A line L in $\mathbb{R}^3$ is given by $X = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$. Compute its point of intersection with the plane E from the previous item. **[4 marks]**

(c) Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for an algebra of vectors V, and let $\vec{w}$ be an arbitrary vector in V.

(i) When do we say that scalars $a_1, a_2, \ldots, a_n$ are the *coordinates* of $\vec{w}$ with respect to B? **[1 mark]**

(ii) Prove that the coordinates of $\vec{w}$ with respect to B are uniquely determined. **[3 marks]**

Question 2 [SAT & Predicate Logic]

(a) (i) Let $p_0, p_1, q_0, q_1, r_0, r_1$ be atoms capturing the states of three cells called p, q, and r, that can each either hold a 0 or a 1: p_i captures the fact that cell p

holds the value i, and similarly for the other atoms. Consider the following formula:

$$(p_0 \vee p_1) \wedge (q_0 \vee q_1) \wedge (r_0 \vee r_1) \wedge (\neg p_0 \vee \neg p_1) \wedge (\neg q_0 \vee \neg q_1) \wedge (\neg r_0 \vee \neg r_1)$$
$$\wedge (p_0 \vee q_0 \vee r_0) \wedge (p_1 \vee q_1) \wedge (p_1 \vee r_1) \wedge (q_1 \vee r_1)$$

Using DPLL, prove whether the above formula is satisfiable or not. Detail your answer. What property of the three cells p, q, and r, is this formula capturing?

[4 marks]

(ii) Given a CNF (Conjunctive Normal Form) that contains a clause composed of a single literal, can it be proved using Natural Deduction? Justify your answer.

[2 marks]

(b) Consider the following domain and signature:

- Domain: $\mathbb{N}$
- Function symbols: `zero` (arity 0); `succ` (arity 1); $*$ (arity 2)
- Predicate symbols: `even` (arity 1); `odd` (arity 1); $=$ (arity 2)

We will use infix notation for the binary symbols $*$ and $=$. Consider the following formulas that capture properties of the above predicate symbols:

- let S_1 be $\forall x.(\texttt{even}(x) \rightarrow \exists y.x = 2 * y)$
- let S_2 be $\forall x.((\exists y.x = \texttt{succ}(2 * y)) \rightarrow \texttt{odd}(x))$
- let S_3 be $\forall x.\forall y.(x = y \rightarrow \texttt{succ}(x) = \texttt{succ}(y))$

where for simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc.

(i) Provide a constructive Sequent Calculus proof of:

$$S_1, S_2, S_3 \vdash \forall x.(\texttt{even}(x) \rightarrow \texttt{odd}(\texttt{succ}(x)))$$

[6 marks]

(ii) Provide a model M such that $\vDash_M \forall x.(\texttt{even}(x) \rightarrow \texttt{odd}(\texttt{succ}(x)))$ **[2 marks]**

(iii) Provide a model M such that $\neg \vDash_M \forall x.(\texttt{even}(x) \rightarrow \texttt{odd}(\texttt{succ}(x)))$ **[2 marks]**

(c) Let p be a predicate symbol of arity 1 and q be a predicate symbol of arity 2. Let F be the Predicate Logic formula $(\forall x.(p(x) \wedge \exists y.q(x, y))) \rightarrow \forall x.\exists y.(p(x) \wedge q(x, y))$. Provide a constructive Natural Deduction proof of F. You are not allowed to make use of further assumptions so all your hypotheses should be canceled in the final proof tree. **[4 marks]**