Mathematical and Logical Foundations of Computer Science
Class Test

Solutions

Resit Class Test 2020/21

Mathematical and Logical Foundations of Computer Science
Class Test

Question 1 [Math]

(a) **Set Theory**

(i) Consider the binary relation R on $\mathbb{Z}_5$ given by

$$R = \{(x, y) \in \mathbb{Z}_5 \times \mathbb{Z}_5 \mid x + y = 0\}$$

List the elements of R. **[2 marks]**

Is R reflexive, symmetric, transitive? Is it an equivalence relation? Give brief justifications or counterexamples. **[4 marks]**

(ii) Discuss the similarities and differences between $\mathbb{Z}$ and the elements of the Java datatype `int`. **[4 marks]**

(b) **Linear Algebra**

Consider the following points in 3D space

$$P_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad P_3 = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$$

(i) Show that they form a *scalene* triangle (i.e., a triangle in which no two sides have the same length). **[3 marks]**

(ii) The triangle defines a plane E in $\mathbb{R}^3$. Give its parametric representation and its normal form. **[3 marks]**

(iii) Find the nearest neighbour of $Q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ on E. **[2 marks]**

(iv) Find a basis for $\mathbb{R}^3$ containing two sides of the triangle. **[2 marks]**

Model answer / LOs / Creativity:

(a) (i) $R = \{(0, 0), (1, 4), (2, 3), (3, 2), (4, 1)\}$. R is

- not reflexive because $(1, 1) \notin R$;
- symmetric because $x + y = y + x$ in $\mathbb{Z}_5$;
- not transitive because $(2, 3)$ and $(3, 2)$ belong to R but $(2, 2)$ does not;
- not an equivalence relation because it is neither reflexive nor transitive.

(ii) Some properties discussed in the course:

- $\mathbb{Z}$ has infinitely many elements, `int` has 2^{32} many.
- On both $\mathbb{Z}$ and `int` addition, subtraction, and multiplication are defined.
- Both $\mathbb{Z}$ and `int` satisfy the ring laws.
- $\mathbb{Z}$ satisfies the cancellation law for multiplication, `int` does not.

(Each of these, if properly stated, may attract two points.)

(b) (i) We compute the length of each side:

$$|\overrightarrow{P_1P_2}| = \sqrt{(0-1)^2+(1-0)^2+(1-2)^2} = \sqrt{1+1+1} = \sqrt{3}$$
$$|\overrightarrow{P_2P_3}| = \sqrt{(-1-0)^2+(-3-1)^2+(2-1)^2} = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$$
$$|\overrightarrow{P_3P_1}| = \sqrt{(1-(-1))^2+(0-(-3))^2+(2-2)^2} = \sqrt{4+9+0} = \sqrt{13}$$

and find that they are all different.

(ii) One parametric representation is $X = \begin{pmatrix}1\\0\\2\end{pmatrix} + s\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix} + t\cdot\begin{pmatrix}-1\\-4\\1\end{pmatrix}$, using a corner and two sides of the triangle.

A normal to the plane, according to the formula in the handout is given by $\vec{n} = \begin{pmatrix}-3\\2\\5\end{pmatrix}$. Using a corner of the triangle we get the normal form $-3x_1 + 2x_2 + 5x_3 = 7$.

(iii) Using the formula in the handout we compute the nearest neighbour as $Q' = Q + \frac{7}{38}\cdot\begin{pmatrix}-3\\2\\5\end{pmatrix} = \frac{7}{38}\cdot\begin{pmatrix}-3\\2\\5\end{pmatrix}$.

(iv) We can choose two sides of the triangle plus the normal: $B = \{\begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-4\\1\end{pmatrix}, \begin{pmatrix}-3\\2\\5\end{pmatrix}\}$.

(Because the normal is at a right angle to the triangle, it is linearly independent of the sides of the triangle.)

Learning outcomes: "Solve mathematical problems in algebra and set theory" (a & b); "Apply mathematical techniques to solve a problem within a computer science setting" (a-i).

Question 2 [Logic]

(a) **Propositional logic**

(i) Let F be the following proposition: $(\neg(P \vee Q)) \to ((\neg P) \to (\neg Q) \to R) \to R$. Provide an intuitionistic Natural Deduction proof of F. **[5 marks]**

(ii) Is F valid? Justify your answer. **[1 mark]**

(iii) Consider the formula F, where $\vee$ is turned into an $\wedge$, i.e., let G be $(\neg(P \wedge Q)) \to ((\neg P) \to (\neg Q) \to R) \to R$. Is G valid? Is it satisfiable? Justify your answers. **[4 marks]**

(b) **Predicate Logic**

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: < (arity 2)

We will use infix notation for the binary symbol $<$. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\forall x. x < \mathtt{succ}(x)$
- let S_2 be $\forall x. \neg(x < x)$

(i) Provide a constructive Sequent Calculus proof of:

$$S_1, S_2 \vdash \neg(\forall x.\forall y. x < y \to \mathtt{succ}(x) < y)$$

You are allowed to write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc. (Hint: 9 proof steps are enough to complete this proof, but feel free to use a different number of proof steps in your proof.) **[6 marks]**

(ii) Provide a model M_1 such that $\vDash_{M_1} \neg(\forall x.\forall y. x < y \to \mathtt{succ}(x) < y)$, and a model M_2 such that $\vDash_{M_2} \forall x.\forall y. x < y \to \mathtt{succ}(x) < y$ **[4 marks]**

Model answer / LOs / Creativity:

(a) (i) Here is an intuitionistic Natural Deduction proof of F:

$$
\begin{prooftree}
\AxiomC{$\overline{\neg P \to \neg Q \to R}^{\,2}$}
\AxiomC{$\overline{\neg(P \vee Q)}^{\,1}$}
\AxiomC{$\overline{P}^{\,3}$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$P \vee Q$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{3 $[\neg I]$}
\UnaryInfC{$\neg P$}
\RightLabel{$[\to E]$}
\BinaryInfC{$\neg Q \to R$}
\AxiomC{$\overline{\neg(P \vee Q)}^{\,1}$}
\AxiomC{$\overline{Q}^{\,4}$}
\RightLabel{$[\vee I_R]$}
\UnaryInfC{$P \vee Q$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{4 $[\neg I]$}
\UnaryInfC{$\neg Q$}
\RightLabel{$[\to E]$}
\BinaryInfC{R}
\RightLabel{2 $[\to I]$}
\UnaryInfC{$((\neg P) \to (\neg Q) \to R) \to R$}
\RightLabel{1 $[\to I]$}
\UnaryInfC{$(\neg(P \vee Q)) \to ((\neg P) \to (\neg Q) \to R) \to R$}
\end{prooftree}
$$

(ii) By consistency of Natural Deduction, because F is provable it is also valid.

(iii) G is not valid because if P is true and both Q and R are false, then G is also false. It is however satisfiable because if P, Q, and R are all false, then G is true.

(b) (i)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{[Id]}
\UnaryInfC{$0 < 1, S_2 \vdash 0 < 1$}
\RightLabel{[$\forall L$]}
\UnaryInfC{$S_1, S_2 \vdash 0 < 1$}
\AxiomC{}
\RightLabel{[Id]}
\UnaryInfC{$S_1, 1 < 1 \vdash 1 < 1$}
\RightLabel{[$\neg L$]}
\UnaryInfC{$S_1, \neg(1 < 1), 1 < 1 \vdash \perp$}
\RightLabel{[$\forall L$]}
\UnaryInfC{$S_1, S_2, 1 < 1 \vdash \perp$}
\RightLabel{[$\rightarrow L$]}
\BinaryInfC{$S_1, S_2, 0 < 1 \rightarrow 1 < 1 \vdash \perp$}
\RightLabel{[$\forall L$]}
\UnaryInfC{$S_1, S_2, \forall y.0 < y \rightarrow 1 < y \vdash \perp$}
\RightLabel{[$\forall L$]}
\UnaryInfC{$S_1, S_2, \forall x.\forall y.x < y \rightarrow \mathtt{succ}(x) < y \vdash \perp$}
\RightLabel{[$\neg R$]}
\UnaryInfC{$S_1, S_2 \vdash \neg(\forall x.\forall y.x < y \rightarrow \mathtt{succ}(x) < y)$}
\end{prooftree}
$$

(ii) The model $M_1 = \langle \mathbb{N}, \langle 0, +1 \rangle, \langle \{ \langle n, m \rangle \mid n < m \} \rangle \rangle$, where $+1$ is the function that given a number increments it by 1, is a model of $\neg(\forall x.\forall y.x < y \rightarrow \mathtt{succ}(x) < y)$, and the model $M_2 = \langle \mathbb{N}, \langle 0, +1 \rangle, \langle \{ \langle n, m \rangle \mid \text{True} \} \rangle \rangle$ is a model of $\forall x.\forall y.x < y \rightarrow \mathtt{succ}(x) < y$

Learning outcomes: "Understand and apply algorithms for key problems in logic such as satisfiability." (a); "Write formal proofs for propositional and predicate logic" (a & b); "Apply logical techniques to solve a problem within a computer science setting" (a & b).