UNIVERSITY OF BIRMINGHAM

School of Computer Science

Mathematical and Logical Foundations of Computer Science Class Test

Resit Class Test 2020/21

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Class Test

Question 1 [Math]

(a) **Set Theory**

(i) Consider the binary relation R on $\mathbb{Z}_5$ given by

$$R = \{(x, y) \in \mathbb{Z}_5 \times \mathbb{Z}_5 \mid x + y = 0\}$$

List the elements of R. **[2 marks]**

Is R reflexive, symmetric, transitive? Is it an equivalence relation? Give brief justifications or counterexamples. **[4 marks]**

(ii) Discuss the similarities and differences between $\mathbb{Z}$ and the elements of the Java datatype `int`. **[4 marks]**

(b) **Linear Algebra**

Consider the following points in 3D space

$$P_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad P_3 = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$$

(i) Show that they form a *scalene* triangle (i.e., a triangle in which no two sides have the same length). **[3 marks]**

(ii) The triangle defines a plane E in $\mathbb{R}^3$. Give its parametric representation and its normal form. **[3 marks]**

(iii) Find the nearest neighbour of $Q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ on E. **[2 marks]**

(iv) Find a basis for $\mathbb{R}^3$ containing two sides of the triangle. **[2 marks]**

Question 2 [Logic]

(a) **Propositional logic**

(i) Let F be the following proposition: $(\neg(P \vee Q)) \rightarrow ((\neg P) \rightarrow (\neg Q) \rightarrow R) \rightarrow R$. Provide an intuitionistic Natural Deduction proof of F. **[5 marks]**

(ii) Is F valid? Justify your answer. **[1 mark]**

(iii) Consider the formula F, where $\vee$ is turned into an $\wedge$, i.e., let G be $(\neg(P \wedge Q)) \rightarrow ((\neg P) \rightarrow (\neg Q) \rightarrow R) \rightarrow R$. Is G valid? Is it satisfiable? Justify your answers. **[4 marks]**

(b) **Predicate Logic**

Consider the following signature:

- Function symbols: $\texttt{zero}$ (arity 0); $\texttt{succ}$ (arity 1)
- Predicate symbols: < (arity 2)

We will use infix notation for the binary symbol $<$. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\forall x.x < \texttt{succ}(x)$
- let S_2 be $\forall x.\neg(x < x)$

(i) Provide a constructive Sequent Calculus proof of:

$$S_1, S_2 \vdash \neg(\forall x.\forall y.x < y \rightarrow \texttt{succ}(x) < y)$$

You are allowed to write 0 for $\texttt{zero}$, 1 for $\texttt{succ}(\texttt{zero})$, 2 for $\texttt{succ}(\texttt{succ}(\texttt{zero}))$, etc. (Hint: 9 proof steps are enough to complete this proof, but feel free to use a different number of proof steps in your proof.) **[6 marks]**

(ii) Provide a model M_1 such that $\vDash_{M_1} \neg(\forall x.\forall y.x < y \rightarrow \texttt{succ}(x) < y)$, and a model M_2 such that $\vDash_{M_2} \forall x.\forall y.x < y \rightarrow \texttt{succ}(x) < y$ **[4 marks]**